MAGNETOHYDRODYNAMIC DISTURBANCES IN A VISCOUS FLUID OF FINITE ELECTRICAL CONDUCTIVITY
(Unsymmetric Case)

A Thesis Submitted in Partial Fulfilment of the
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By

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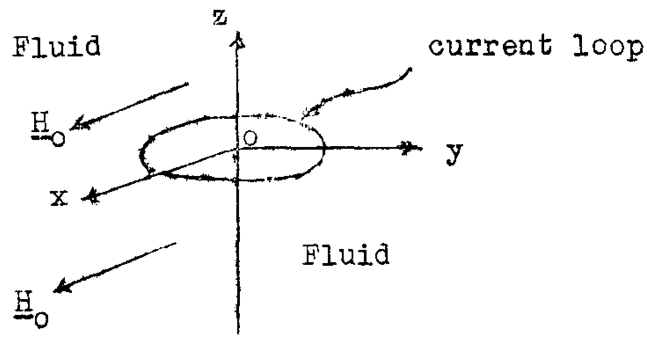
SUMMARY

It is supposed that we have initially an INFINITE mass at rest of a VISCOUS fluid of finite electrical conductivity in which a uniform field $\underline{H}_o$ is prevailing. It is supposed then a dipole of moment $\underline{M}$ is suddenly introduced in the above configuration of the system. As a result of this sudden introduction a velocity $\underline{U}$ is produced in a certain volume of the fluid as well as a small disturbance $\underline{h}$ of $\underline{H}_o$. The dipole, whose axis is always taken in the Z-direction, can be replaced by a small current loop in the (x,y)-plane, of strength $\underline{I}$, radius a, with its centre always fixed at the origin of coordinates, and such that

$$\underline{M} = \lim_{a \to o} \pi a^2 \underline{I} \quad \dots \quad \dots \qquad (1)$$

$$\underline{I} = \underline{1}_\phi I \, \delta(\rho - a). \, H(t) \quad \dots \qquad (2)$$

where $\underline{1}_\phi$ is a unit vector in the direction of ϕ increasing (ρ, ϕ, Z) are cylindrical polar coordinates, H(t) is the Heaviside unit function, and δ is Dirac's delta function. Only the UNSYMMETRIC case, where the uniform field $\underline{H}_o$ is always taken parallel to the x-axis, perpendicular to the dipole axis, is dealt with.



UNSYMMETRIC CASE

Uniform field perpendicular to the dipole axis

In chapter I of this work boundary conditions on crossing the loop have been introduced in the basic equattions used, namely Maxwell's equations and the hydrodynamic equations.

In chapter II, mathematical solutions representing the behaviour of both $\underline{U}$ and $\underline{h}$ have been re-studied in the case where a MAGNETIC dipole has been suddenly introduced. Extensive figures calculated from the obtained mathematical results have been drawn. The behaviour of the phenomena has been traced at different long intervals of time.

It is worth to note here that the figures so obtained, in the case of a VISCOUS fluid, show that the effect of viscosity in the case of an infinite fluid, is of no great importance. In fact, it is found that there is a great similarity between the behaviour of both U and h in this case and those obtained by MORANDIS* (1959) in a non-viscous fluid.

In chapter III the same problem has been valued for a viscous fluid, but with the sudden introduction of an OSCILLATING dipole instead of a magnetic one. This case has been already published as a joint work in IL NUOVO CIMENTO (1969), Series X, Vol. 59B, pp. 1-11. It is seen that the solution of the problem of an oscillating dipole is much more difficult than that of a magnetic one. A new tedious effort is needed to represent the obtained mathematical results in a form suitable for computation.

It should be noted here that the problem in the SYMMETRIC case where H_o is taken parallel to the dipole axis is now under consideration by Mrs. BAGHDAD FAHMY and will be submitted in a future date.

CHAPTER I

DIPOLE IN A VISCOUS FLUID

§ 1.1 FUNDAMENTAL EQUATIONS

The phenomena can be described by using Maxwell's equations and the hydrodynamic equation. Maxwell's equations in this case and in the presence of a moving electrically conducting matter with velocity $\underline{U}$ and electrical conductivity σ are :

$$\text{curl } \underline{E} = -\partial \underline{H} / \partial t \quad \ldots \quad \ldots \qquad (3)$$

$$\text{curl } \underline{H} = 4 \quad \underline{J} \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

where

$$\underline{J} = \sigma \; (\underline{E} + \underline{\mathcal{E}} + \underline{U} \wedge \underline{H}) \quad \ldots \quad \ldots \qquad (5)$$

$$\sigma \mathcal{E} = \underline{1}_{\phi} \; I \; \delta(\rho - a) \, . \, H(t) \quad \ldots \qquad (6)$$

also

$$\text{div } \underline{H} = 0 \quad \ldots \quad \ldots \quad \ldots \qquad (7)$$

$$\text{div } \underline{J} = 0 \quad \ldots \quad \ldots \quad \ldots \qquad (8)$$

where $\underline{E}$ is the electric field, $\underline{J}$ the electric current and $\underline{H}$ is the magnetic field. The effect of the disturbing source has been expressed by in introducing the electromotive force $\mathcal{E}$ of the current loop in equation (3).

Making the substitution

$$\underline{H} = \underline{H}_0 + \underline{h} \quad \ldots \quad \ldots \qquad (9)$$

where $$\underline{h} \ll \underline{H}_0 \quad \ldots \quad \ldots \quad \ldots \tag{10}$$

$$\text{div } \underline{h} = 0 \quad \ldots \quad \ldots \tag{11}$$

$$\underline{H}_0 = H_0 \, \underline{1}_x \quad \ldots \quad \ldots \tag{12}$$

where $\underline{1}_x$ is a unit vector in the direction of x. Taking the curl of equations (4) and (5), substituting in equation (3), applying conditions (7) and (8), we see on neglecting squares and products of $\underline{H}$ and $\underline{h}$, that

$$\left(\frac{\partial}{\partial t} - \frac{1}{4\pi\sigma} \nabla^2\right)\underline{h} = H_0 \text{ grad } U + 4\pi \text{ curl } \underline{I} \tag{13}$$

The hydrodynamic equation is

$$\rho \frac{d\,\underline{U}}{d\,t} = \underline{F} - \nabla p + \rho\,\nu \nabla^2 \underline{U} \quad \ldots \tag{14}$$

where the moving matter is considered to be incompressible, with density ρ, hydrostatic pressure p, and kinematic viscosity ν.

Also the equation of continuity shows that

$$\text{div } \underline{U} = 0 \ldots \quad \ldots \quad \ldots \tag{15}$$

The Pondermotive force acting on the loop is

$$\underline{F} = \underline{J} \wedge \underline{H} - \underline{I} \wedge \underline{H} \quad \ldots \quad \ldots \quad \ldots \tag{16}$$

The effect of the disturbance is expressed by the force $-\underline{I} \wedge \underline{H}$ to balance the Lorentz force on the loop, thus preventing its motion.

Substituting from (9) and (16) in equation (14), we see on neglecting squares and products of $\underline{U}$ and $\underline{h}$ that

$$\frac{d\underline{U}}{dt} = \frac{H_o}{4\pi\rho}\frac{\partial \underline{h}}{\partial x} - \text{grad}\,\frac{1}{\rho}\left[p + \frac{1}{8\pi}(\underline{H}_o+\underline{h})^2\right] + \nu\nabla^2\underline{U} \quad \ldots \quad (17)$$

Putting

$$\underline{P} = -\,\text{grad}\,\check{\omega} \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where

$$\check{\omega} = \frac{1}{\rho}\left[p + \frac{1}{8\pi}(\underline{H}_o + \underline{h})^2\right], \quad \ldots \quad (19)$$

taking the divergence of equation (17) and using conditions (11) and (15), we find that

$$\text{div}\,\underline{P} = 0 \quad \ldots \quad \ldots \quad (20)$$

It is found more convenient to express equations (13) and (17) in non-dimensional units. This can be done by introducing new variables defined by

$$\left.\begin{array}{l} \underline{h} = H_o\underline{h}', \quad \underline{U} = V\underline{U}', \quad \underline{P} = \omega_o V\underline{P}', \quad t' = \omega_o t, \\ \underline{r} = \omega_o\underline{r}'/V, \quad \nu = \nu' V^2/\omega_o, \end{array}\right\} \quad (21)$$

where V is the Alfvén velocity defined by

$$V^2 = H_o^2/4\pi\rho \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

and ω_o is a frequency defined by

$$\omega_o = \sigma H_o^2/\rho. \quad \ldots \quad \ldots \quad (23)$$

Hence forth the primes will be omitted.

The fundamental equations (13) and (17) can now be expressed in the following non-dimensional form:

$$(\partial/\partial t - \nabla^2)\underline{h} = \partial \underline{U}/\partial x + \text{curl } \underline{I} \quad \ldots \qquad (24)$$

$$\partial \underline{U}/\partial t = \partial \underline{h}/\partial x + \underline{1}_x \wedge \underline{I} + \underline{P} + \nu \nabla^2 U \qquad (25)$$

where

$$\underline{P} = -\text{grad } \frac{1}{\rho}\left[p + \frac{1}{8\pi}(\underline{1}_x + \underline{h})^2\right] \ldots \qquad (26)$$

Introducing the solenoidal vectors $\underline{\psi}$, $\underline{\chi}$ and $\underline{\zeta}$ such that

$$\underline{h} = \text{curl } \psi, \quad \underline{U} = \text{curl } \chi \qquad (27)$$

we see that equations (24) and (25) respectively can be put in the form

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\underline{\psi} = \frac{\partial \chi}{\partial x} + \underline{I} \quad \ldots \quad \ldots \qquad (28)$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\underline{\chi} = \frac{\partial \psi}{\partial x} + \underline{B} \quad \ldots \quad \ldots \qquad (29)$$

where

$$\text{curl } \underline{B} = \underline{1}_x \wedge \underline{I}_\phi + \underline{P} \quad \ldots \quad \ldots \quad \ldots \qquad (30)$$

which implies that

$$\nabla^2 \underline{B} = \underline{1}_\phi \frac{\partial I}{\partial x} \quad \ldots \quad \ldots \qquad (31)$$

§ 1.2 INTEGRAL TRANSFORMS USED

1. The Heaviside transform denoted by broken brackets around the transformed symbol. It is defined by

$$\langle \psi(\underline{r}, p) = p \int_0^\infty e^{-pt} \psi(\underline{r},t)dt \quad \ldots \qquad (32)$$

Integrating by parts shows that

$$\langle \frac{\partial \psi}{\partial t} \rangle = p(\langle \psi \rangle - \psi_o), \ \langle \frac{\partial^2 \psi}{\partial t^2} \rangle = p^2(\langle \psi \rangle - \psi_o) - p\psi_1$$

where

$$\psi_o = \psi\ (r,o) \quad , \qquad \psi_1 = (\frac{\partial \psi}{\partial t})_{t=o}$$

ψ_o and ψ_1 will be neglected in our results, since initially the field is uniform and the fluid is at rest.

2. The double Fourier transform denoted by a superscript star, thus:

$$\psi^*(\ell,m,z,t) = \int_{-\infty}^{\infty}\int \psi\ (x,y,z,t)\ \exp\left[i(\ell x+my)\right] dxdy \quad \ldots \quad (33)$$

the solution to which is

$$\psi(x,y,z,t) = \frac{1}{4\pi^2} \int^{\infty}\int_{-\infty} \psi^*(\ell,m,z,t)\exp\left[-i(\ell x+my)\right] d\ell dm \quad \ldots \ (34)$$

3. The Fourier transform denoted by a superscript dagger, thus:

$$\psi^{+}(x,y,n,t) = \int_{-\infty}^{\infty} \psi(x,y,z,t)\, e^{inz}\, dz \qquad (35)$$

the solution to which is

$$\psi(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^{+}(x,y,n,t)\, e^{-inz}\, dn \qquad (36)$$

§ 1.3 TO DEFINE I^{*}

Let I_x and I_y be the x and y components of $\underline{I}$ as defined in the summary by

$$\underline{I} = \underline{1}_{\phi}\, I\, \delta(\rho - a).H(t) \;;$$

then

$$I_x = -\,\underline{I}\sin\phi \;, \qquad I_y = \underline{I}\cos\phi \quad \ldots \qquad (37)$$

Let $\ell = \xi\cos\alpha \;, \qquad m = \xi\sin\alpha$

then $\qquad \xi^2 = \ell^2 + m^2 \qquad \ldots \quad \ldots \quad \ldots \qquad (38)$

and $\ell x + m y = \rho\,\xi\cos(\phi - \alpha)$

also

$$I_x^{*} = -i\underline{I}^{*}\sin\alpha, \quad I_y^{*} = i\underline{I}^{*}\cos\alpha \quad \ldots \quad \ldots \qquad (39)$$

It then follows that

$$\underline{I}^{*} = i\, I_x^{*}\sin\alpha - i\, I_y^{*}\cos\alpha$$

Applying equation (33) to equations (37) and substituting the obtained values for I_x^{*} and I_y^{*} in equation (38), we have

$$\underline{I}^{*} = -i \int\!\!\int_{-\infty}^{\infty} I\cos(\phi-\alpha)\delta(\rho - a)e^{i(\ell x+my)}\, dxdy$$

$$= 2\,\rho\, a\, I\, J_1(a\,\xi)$$

where $J_1(z)$ denotes the Bessel function of the first kind of order unity. Making the substitution

$$\underline{M} = \lim_{a \to o} \pi a^2 \underline{I} \,,$$

we have

$$\underline{I}^* = \xi \underline{M} \,. \quad \ldots \quad \ldots \quad \ldots \qquad (40)$$

Then equations (37) show that

$$I_x^* = - imM \,, \quad I_y^* = i \ell M \quad \ldots \quad \ldots \qquad (41)$$

§ 1.4 PROCEDURE FOR OBTAINING A FORMAL SOLUTION

Applying the Heaviside transform as given in equation (32), the star and dagger Fourier transforms as shown in relations (33) and (35) respectively, equations (28) and (29) show respectively that

$$(p + \xi^2 + n^2) \langle \underline{\psi}^{*+} \rangle = -i\ell \langle \underline{\chi}^{*+} \rangle + \underline{1}_\phi \, I^* \qquad (42)$$

$$(p + \gamma \xi^2 + \gamma n^2) \langle \underline{\chi}^{*+} \rangle = -i\ell \langle \underline{\psi}^{*+} \rangle + \langle \underline{B}^{*+} \rangle \qquad (43)$$

where

$$\langle \underline{B}^{*+} \rangle = \frac{i \ell}{\xi^2 + n^2} \, I^* \, \underline{1}_\phi \,, \quad \ldots\ldots \qquad (44)$$

so that

$$\langle \underline{B}^* \rangle = \frac{i \ell}{2 \xi} \, I^* \, \underline{1}_\phi \, e^{-\xi |z|} \quad \ldots \qquad (45)$$

Making use of result (44), solving equations (42) and (43) simultaneously, we have

$$\left\langle \underline{\psi}^{*+} \right\rangle = I^{*} \frac{p\,(\xi^2+n^2) + \gamma(\xi^2+n^2) + \ell^2}{(\xi^2+n^2)\left[p^2+p\mu\,(\xi^2+n^2)+\gamma(\xi^2+n^2)+\ell^2\right]}\, 1_{\phi} \qquad (46)$$

$$\left\langle \underline{\chi}^{*+} \right\rangle = I^{*} \frac{i\,\ell\, p}{(\xi^2+n^2)\left[p^2+p\mu\,(\xi^2+n^2)+\gamma(\xi^2+n^2)+\ell^2\right]}\, 1_{\phi} \qquad (47)$$

where $\mu = 1 + \gamma$ ... ... (48)

It is to be noticed that

$$\psi_z^{*+} = 0 \;, \quad \chi_z^{*+} = 0$$