BASES IN BANACH SPACES

THESIS

Submitted in Partial Fulfilment
Of The Requirments
For The Award
Of
The (M. Sc.) Degree
In
(Pure Mathematics)

By

Mohamed Sabri Salem Ali
Demonstrator in Mathematics Department,
Faculty of Education,
Ain Shams University.
(B. Sc. 1985)

Supervised By

Professor
Dr. Entisarat El-Shobaky
Faculty of Science
Ain Shams University

Associate Professor
Dr. Nashat Faried
Faculty of Science
Ain Shams University

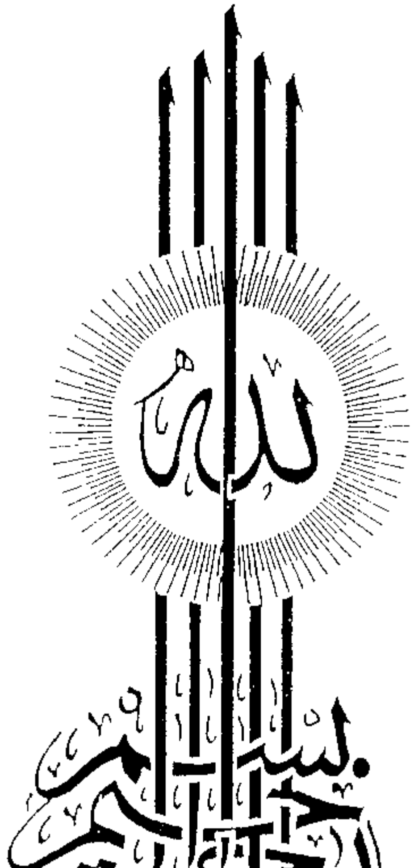
Submitted To

Department of Mathematics
Faculty of Science
Ain Shams University

(CAIRO)
1993

بسم الله الرحمن الرحيم

وقل ربِّ زدني علماً


بسم الله الرحمن الرحيم

ACKNOWLEDGMENT

My deep appreciation and sincere gratitude are presented to my supervisor, prof. Dr. **Entisarat El-Shobaky,** *professor of pure Mathematics, Faculty of science, Ain Shams University, for suggesting the investigations culminating in this work, for constructive discussion, for her help and encouragement and for providing moral support.*

I would like to acknowledge my deepest gratitude and sincere thanks to my supervisor, Dr. **Nashat Faried Mohamed,** *associate professor of pure Mathematics, Faculty of science, Ain Shams University, for following the interesting point of research, tremendous efforts, continuous supervision and constant encouragement and for his facilities provided to me during the preparation of this thesis and invaluable advice.*

THE M. Sc. COURSES

Studied By The Author During The Period

(September 1985 - September 1986)

(At Ain Shams University, Faculty of Science)

	Hours/Week
1. **Graph Theory.**	2
2. **Syntax and Semantics.**	2
3. **Programming Development.**	2
4. **Mathematical Programming.**	2
5. **Computability and Complexity.**	2

CONTENTS

INTRODUCTION

The purpose of this thesis is to give a survey on some basic facts pertaining to bases for Banach spaces. The notion of a basis of a Banach space was introduced by J. Schauder [42].

The theory of bases in Banach spaces was developed substantially, since the appearance of Singer's monograph [45].

The fact that in the common Banach spaces there exists a basis led J. Schauder to pose the fundamental question of his paper [42] in 1927, and which was explicitly mentioned in Banach monograph ([2],P. 111) in 1932 about *whether every separable Banach space has a basis*. This problem (known as the *basis problem*) remained open for a long time and was only solved in the negative by P. Enflo [13] in 1973. In connection with this problem, A. Grothendieck ([17], PP. 164-191) has defined the approximation property and raised the question, known as the *approximation problem, whether every separable Banach space has the approximation property*. In fact, P. Enflo [13] has constructed a subspace of c_o which does not have the bounded approximation property.

In this thesis, we study the above mentioned problems and properties of bases in Banach spaces. Basically we follow the work of Bessaga and Pelczynski [4] about a principle for selecting basic sequences, which are basis in the closed linear set spanned upon them. We study the work of Morrell and Retherford [32] which outlines a technique for constructing basic sequences. Besides, for completeness of this task, we shall study a modified version due to A. Szankowski [50] of that used by P. Enflo [13] in his original solution of the approximation problem.

Finally, we study Banach spaces having unconditional bases and as an application we given in [14] a two sided estimation for the approximation numbers of multiplication operators in these spaces. The obtained result is compatible with a known result mentioned in [37] for the approximation numbers of multiplication operators in ℓ_p space (which have of course unconditional basis).

The thesis consists of four chapters.

CHAPTER 0:

In this chapter we give a short introduction to the working tools from functional analysis which are really needed for the development of different topics of this thesis. Moreover, we omit proofs of all the lemmas, theorems and corollaries given there.

CHAPTER I:

In this chapter we **study** basic definitions and properties of bases in Banach spaces such as: The continuity of the coefficient functionals associated to the basis of a Banach space X, the equivalence of strong and weak bases for X and the criterion for checking whether a given sequence is a basis for X. At the end of this chapter, we consider the work of Bessaga and Pelczynski [4] which establishes the fundamental criterion for selection basic sequences.

CHAPTER II:

This chapter is devoted to studying some properties of sequences in infinite dimensional Banach spaces. We consider the work of Morrell and Retherford [32] in which they use a high technique to construct basic sequences. Also,we present Enflo 's solution of the approximation problem or, more precisely, a modified version of it, due to A. Szankowski [50].

CHAPTER III:

Finally, this chapter is devoted to studying some properties of the approximation numbers and unconditional bases to be used in our work (Theorem (3.9)).

In [14] we give an estimation of the approximation numbers of a multiplication operator on any arbitrary Banach space with an unconditional basis. Under certain condition, satisfied by the space ℓ_p $(1 \le p \le \infty)$, we obtain the result of [37].

CHAPTER 0

CHAPTER O

PRELIMINARY CONCEPTS

In the four paragraphs of this chapter we present some basic definitions and facts from functional analysis. These preliminaries will be used in the subsequent chapters but from time to time we supplement them with other results which make the discussion more complete. As a rule we shall work with real scalars. Furthermore, we omit proofs of all the lemmas, theorems and corollaries given here. Finally, we intend to preserve the notation and terminology of M. M. Day [8], N. Dunford and J. T. Schwartz [11], E. Kreyszig [25], and I. Singer ([45], [46]).

0.1. Notations

We list up basic notations to be used in the sequel:

$\mathbb{R}$ is the field of real numbers.

$\mathbb{N}$ is the set of natural numbers.

Π = the set of all permutations of the set $\mathbb{N}$.

$\mathcal{D} = \{\{i_1,\dots,i_n\}\subset\mathbb{N}:\ 1\le n<\infty\}$ is a countable directed set (by inclusion) of all finite subsets of $\mathbb{N}$.

$\mathcal{O} = \{(i_r)\subset\mathbb{N}:\ i_1<i_2<\dots\}$ is the set of all increasing sequences of natural numbers.

For any subset F of $\mathbb{N}$, by card (F) we denote the cardinal number of the set F.

δ_{ij} is, as usual, Kronecker's delta.

From now on, unless the contrary is explicitly stated, X and Y denote real Banach spaces. By Banach space we shall mean infinite-dimensional Banach space.

- *For two Banach spaces* X *and* Y*:*

We denote by L(X, Y) the Banach space of all continuous (bounded) linear operators of X into Y endowed with the usual norm.

$$\| T \| = \sup\{\| T(x) \| : x\in X, \| x \| \leq 1\}.$$

The symbol L(X) will be written for L(X, X).

$\mathcal{F}(X, Y)$ denotes the space of all finite dimensional operators from X into Y.

$\mathcal{K}(X, Y)$ denotes the space of all compact operators from X into Y.

$\mathcal{N}(X, Y)$ denotes the space of all nuclear operators from X into Y.

- *For every operator* $T\in L(X, Y)$

$\mathcal{D}(T) \subset X$ denotes the domain of T.

$\mathcal{R}(T) := \{y\in Y: y=T(x) \text{ for some } x\in\mathcal{D}(T)\}$ denotes the range of T

$\mathcal{N}(T) := \{x\in\mathcal{D}(T): T(x)=0\}$ denotes the null space of T, another word for " null space " is " Kernel " of T.

Unless otherwise stated by "operator" we shall mean bounded linear operator.

- *If X is a Banach space:*

We shall denote by $X^* = L(X, \mathbb{R})$ its dual (conjugate) space.

The identity operator of X is denoted by I_X , and by U_X the unit ball $\{x \in X: \| x \| \leq 1\}$ of X. Instead of I_X and U_X we may simply write I and U, respectively, if there is no danger of confusion.

For any subset M of X, span M denotes the set of all finite linear combinations

$$\sum_{i=1}^{n} \alpha_i y_i \qquad (y_i \in M,\ \alpha_i \in \mathbb{R},\ i=1, \ldots, n;\ n=1, 2, \ldots)$$

and by [M] the closure of span M (i.e $[M] = \overline{\text{span}}\ M$).

The following definitions: separable, reflexive and complete spaces can be found in [25]

0.2. Sequence and Function Spaces

Our objective in this part is to define and state some properties of a special Banach spaces that will be referred to at various places throughout the thesis (c.f [30], [52], [53]).

0.2.1. Sequence Spaces.

These are Banach spaces which can be presented in some natural manner as spaces of sequences. Examples of sequence spaces are Lorentz sequence spaces, modular sequence spaces, Orlicz sequence spaces and other sequence spaces. In the sequel we will concern with the following spaces. (cf [28], [43]).

– Let p be a real number such that $1 \leq p < \infty$. We denote by ℓ_p the space of all sequences $\alpha=(\alpha_i)_{i=1}^{\infty}$ of scalars such that $\sum_{i=1}^{\infty} |\alpha_i|^p<\infty$, with norm defined by

$$\|\alpha\| = \left(\sum_{i=1}^{\infty} |\alpha_i|^p \right)^{\frac{1}{p}}.$$

By ℓ_p^n we denote the space of n-tuples with the above norm.

ℓ_p is a separable weakly sequentially complete Banach space. ℓ_1 is not reflexive and $\ell_1^* = \ell_\infty$.

If $1<p<\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, ℓ_p is reflexive and $\ell_p^* = \ell_q$

- ℓ_∞ denotes the space of all bounded sequences $\alpha = (\alpha_n)$ of scalars with the norm

$$\|\alpha\| = \sup_n |\alpha_n|.$$

ℓ_∞ is a Banach space which is neither separable nor weakly sequentially complete. In addition, ℓ_∞ is not reflexive.

- by c we denote the space of all convergent sequences $\alpha=(\alpha_n)$ of scalars endowed with the norm $\|\alpha\| = \sup_n |\alpha_n|$. c is a separable Banach Space.

- c_o denotes the closed linear subspace of c of all sequences converging to zero. Neither c nor c_o are weakly sequentially complete and both, c and c_o, are not reflexive.

$c^* = c_o^* = \ell_1$ where ℓ_1 is described above.

0.2.2. Function Spaces.

Banach function spaces are Banach spaces of measurable functions in which the norm is related to the underlying measure