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ON THE REDUCTION OF ROW-FINITE MATRICES
AND ITS USE IN SOLVING LINEAR SYSTEMS
OF ALGEBRAIC AND DIFFERENTIAL EQUATIONS

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P R E F A C E

Infinite matrices have received the interest of many mathematicians in different decades. They intervene as tools in many mathematical subjects as well as they are interesting in their own rights. They have been for a long period a useful tool in summability of series as well as in the investigation of basic sets of polynomials. Also they have been, as independent objects, the subject of study. Thus we have the papers of S.A. Ibrahim and R.H. Makar (and his students) on functions, power series and infinite products of infinite matrices. The papers of Vermes and Ayres have investigated the decomposition of row-finite infinite matrices into simple forms of matrices, in particular strings, their results are pure theoretic and do not indicate any applicability. Here in the present thesis, we are interested in the reduction of row-finite infinite matrices to simple forms mostly a kin to those arising in the theory of numerical analysis for finite matrices. The results obtained are applicable in the solution of infinite systems of linear algebraic equations or of linear differential equations. By solving an infinite system we mean that we can obtain any required definite number of the unknowns, whether numbers or functions, provided we are given a sufficient number of equations of the system.

The thesis consists of five chapters. In chapter I, we consider what is so-called a super Hessenberg matrix, i.e., an infinite matrix $\left[a_{ij}\right]$ in which $a_{ij}=0$ for all $j \geq i + m$ where m is a fixed positive integer. A very general theorem is proved on the construction of such a matrix through applying elementary column operations on it, provided the matrix belongs to G(R) the commutative group of row-finite matrices each of which has a unique row-finite reciprocal. The case $m = 3$ has been considered for explanation and for indicating how can the reduction be applied to solve an infinite system of linear equations of which the matrix of coefficients is a super Hessenberg matrix in G(R). The material of this chapter in somewhat abbreviated form, forms paper [12] mentioned in the references at the end of the thesis.

Chapter II deals with row-finite matrices in G(R) which are diagonally dominant by rows and with row - column - finite matrices in G(R) which are diagonally dominant by rows and by columns. Two results are obtained. The interesting feature that the matrix remains diagonally dominant throughout the whole process of reduction is revealed. Emphasis has been made on the fact that there are row-finite matrices (row-column-finite matrices) diagonally dominant by rows (by rows and by columns) which do not belong to G(R). This fact indicates the divergence of the theory of row-finite infinite matrices from that of square matrices. The material of this chapter forms paper [13] .

Chapter III deals with the orthogonal reduction of a row-finite infinite matrix in G(R) into a lower normal matrix. The reducing orthogonal matrix is row-column-finite and is obtained in two different ways. One way is via using Givens' rotation beads and the other is via Householder's transformation beads. The material of this chapter forms paper [14] .

Chapter IV involves further investigation of infinite matrices in G(R). First the reduction to a lower normal form is affected by Gauss elimination method with interchanges. Next the reduction of an upper row-finite matrix in G(R) to an infinite diagonal matrix with non-zero diagonal elements is affected by Gauss elimination method without interchanges. Further we have shown how a real symmetric infinite matrix in G(R), with positive leading minors can be factorized in the form LU where L is a column-finite lower normal matrix and U is the transpose of L , U being in G(R). It has been emphasised that in general a real row-column-finite symmetric matrix with positive leading minors may not belong to G(R). Also the transpose of a matrix in G(R) may not belong to G(R). These facts again indicate the divergence of the theory of row-finite infinite matrices from that of square matrices. The material of this chapter forms paper [15].

In chapter V we first consider the reduction of a row-finite matrix in $G[R(\lambda)]$ and with its leading minors of all orders being non-zero constants into a lower normal matrix with elements in $\mathfrak{Z}[\lambda]$ but with non-zero constant diagonal elements. $\mathfrak{Z}[\lambda]$ is the polynomial domain in λ and $G[R(\lambda)]$ is the commutative group of row-finite matrices with elements in $\mathfrak{Z}[\lambda]$ each matrix having a unique row-finite reciprocal with elements in $\mathfrak{Z}[\lambda]$. We also consider the reduction of a row-column-finite matrix in $G[R(\lambda)]$ and with its leading minors all being non-zero constants, into a diagonal matrix with non-zero constant diagonal elements. We further consider the application of these reductions to the solution of infinite systems of linear differential equations. We consider next the reduction of a semi-block matrix with its elements in $\mathfrak{Z}[\lambda]$ but with its diagonal blocks all having non-zero constant determinants to a lower normal matrix with elements in

$\exists[\lambda]$ and again consider the application of this reduction to the solution of infinite systems of linear differential equations.

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CHAPTER I

ON INFINITE SUPER HESSENBERG MATRICES IN G(R) AND THE SOLUTION OF INFINITE SYSTEMS OF LINEAR EQUATIONS

Abstract :

A theorem is proved on the structure of the so-called an infinite super Hessenberg matrix through performing elementary column operations on it, and the application of this structure to the solution of an infinite system of linear equations is considered.

Consider the system of linear equations

(1) $$Ax = b$$

where $A = [a_{ij}]$ is an infinite matrix such that $a_{ij} = 0$ for all $j > i + m$, $i = 1, 2, 3, \ldots$. Such a matrix is called [11] an infinite super Hessenberg matrix; when $m = 1$, it is an infinite Hessenberg matrix generalizing the finite lower Hessenberg matrix [19]. It has been proved in [8] that an infinite Hessenberg matrix belonging to G(R), the (multiplicative) group of row-finite infinite matrices each of which has a (unique) row-finite reciprocal, must contain an infinite number of zero elements on the (first) superdiagonal, i.e., A is necessarily a lower semi-block matrix. In such a case the system of equations (1) can be simply treated by the well known numerical methods [19] for the solution of finite systems. By such methods we can obtain the values of $x_1, x_2, \ldots, x_N$ for any certain number N of unknowns, and that is what is meant by "solving" the infinite system in (1).

The case $m = 2$ has been considered in [11] and as it is mentioned there it gives a clue to the general case of any m. Indeed it has been proved that when A belongs to $G(R)$, it must contain an infinite number of zero elements on the m th superdiagonal, and this result generalizes that mentioned above for the case $m = 1$. Yet , the case $m = 2$ does not give the general theory which should be aimed at. In this chapter we investigate the case $m = 3$ which leads us to formulate the general theory aimed at.

Consider the matrix

$$(2) \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Assuming that $a_{14} \neq 0$, we perform the elementary column operations

$$\mathrm{col}_j + k_{4j}\,\mathrm{col}_4 \quad , \quad k_{4j} = -\, a_{1j} / a_{14} \quad ,$$

$j = 1, 2, 3$. These operations are equivalent to post-multiplying the matrix A by the infinite unit matrix after inserting the elements k_{41}, k_{42}, k_{43} in the positions (4,1), (4,2), (4,3) respectively. Calling this matrix C_1 and writing A_0 for A, the resulting matrix $A_1 = A_0 C_1$ is such that its elements in the positions (1,1), (1,2), (1,3) are zeros. We write $A_1 = \left[a_{ij}^{(1)}\right]$. Evidently $a_{ij}^{(1)} = a_{ij}$ for all $j \geq 4$, but in general $a_{ij}^{(1)} \neq a_{ij}$ for $j = 1,2,3$.

Assuming that $a_{25}^{(1)}$ ($= a_{25}$) $\neq 0$, we perform the operations

$$\mathrm{col}_j + k_{5j}\,\mathrm{col}_5 \quad , \quad k_{5j} = -\, a_{2j}^{(1)} / a_{25} \quad ,$$

$j = 1, 2, 3, 4$. These operations are equivalent to post-multiplying the matrix A_1 by the infinite unit matrix after inserting the elements k_{51}, k_{52}, k_{53} and k_{54} in the positions (5,1), (5,2), (5,3) and (5,4) respectively. We call this matrix C_2. The resulting matrix $A_2 = A_1 C_2$ is such that the elements in the positions (1,1), (1,2), (1,3); (2,1), (2,2), (2,3), (2,4) are all zeros. Writing $A_2 = \left[a_{ij}^{(2)}\right]$ we have that $a_{ij}^{(2)} = a_{ij}^{(1)} = a_{ij}$, $i = 1,2,3,\ldots$ for all $j \geq 5$. Now $A_2 = A_0\, C_1\, C_2$, where the product matrix $C_1 C_2$ is simply obtained from the infinite unit matrix by inserting the elements $k_{41}, k_{42}, k_{43}; k_{51}, k_{52}, k_{53}, k_{54}$ in the positions (4,1), (4,2), (4,3); (5,1), (5,2), (5,3), (5,4) respectively, i.e., $C_1 C_2$ is simply obtained from C_1 by inserting the elements $k_{51}, k_{52}, k_{53}, k_{54}$ in the positions (5, 1), (5, 2), (5, 3), (5, 4) respectively. Both C_1 and $C_1 C_2$ are simple lower triangular infinite matrices with unit diagonal elements.

Assuming that $a_{i,i+3} \neq 0$ for any i, a continuation of the above treatment shows that there is an infinite lower triangular matrix with unit diagonal elements,

$$C = C_1 C_2 C_3 \dots$$

$$= \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ k_{41} & k_{42} & k_{43} & 1 & & \\ k_{51} & k_{52} & k_{53} & k_{54} & 1 & \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

such that the first three columns of the matrix

$$\bar{A} = AC$$

are zero columns. Thus $\bar{A}$ does not belong to $G(R)$, though both A and C and hence so also AC belong to $G(R)$. This contradiction implies that $a_{i,i+3}$ cannot be non-zero for all $i = 1, 2, 3, \ldots$.

We now assume that n_1 is the first value of i, for which $a_{i,i+3} = 0$. According to the above treatment we have that

$$A_{n_1-1} = A\, C_1\, C_2 \ldots C_{n_1-1}$$

is a matrix in which the elements in row_i before the element $a_{i,i+3}$ are all zeros for $i = 1, 2, \ldots, n_1-1$. We may take C_{n_1} to be the infinite unit matrix itself and write

$$A_{n_1} = A\, C_1 C_2 \ldots C_{n_1-1}\, C_{n_1}$$

where A_{n_1} is the same matrix A_{n_1-1}. Taking $n_1 = 4$, for example, A_4 is of the form

$$A_4 = \begin{bmatrix} o & o & o & x & & & & & \\ o & o & o & o & x & & & & \\ o & o & o & o & o & x & & & \\ x & x & x & x & x & x & o & & \\ x & x & x & x & x & x & x & x & \\ x & x & x & x & x & x & x & x & x \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and $C_1\ C_2\ C_3\ C_4$ is of the form

$$\begin{bmatrix} 1 & & & & & & & \\ o & 1 & & & & & & \\ o & o & 1 & & & & & \\ x & x & x & 1 & & & & \\ x & x & x & x & 1 & & & \\ x & x & x & x & x & 1 & & \\ o & o & o & o & o & o & 1 & \\ o & o & o & o & o & o & o & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Assuming that $a_{n_1+1,n_1+4} \neq 0$, we perform the operations

$$col_j + k_{n_1+4,j}\ col_{n_1+4}\ , \quad k_{n_1+4,j} = -\ a^{(n_1)}_{n_1+1,j}/a_{n_1+1,n_1+4}\ ,$$

$$j = 1,\ 2,\ 3,\ \ldots\ ,\ n_1 + 3.$$

These operations are equivalent to post-multiplying A_{n_1} by the infinite unit matrix after inserting the elements $k_{n_1+4,1}$, $k_{n_1+4,2}$, ... , $k_{n_1+4,\ n_1+3}$ in the positions $(n_1 + 4,1)$, $(n_1 + 4,2)$, ..., $(n_1 + 4,\ n_1 + 3)$. Calling this matrix C_{n_1+1} , then in the resulting matrix

$$A_{n_1+1} = A\ C_1 C_2\ \cdot\ \cdot\ \cdot\ C_{n_1+1}$$

the elements in the i th row before the element $a_{i,i+3}$, $i = 1,\ 2,\ \ldots,\ n_1-1\ ,\ n_1+1$ are all zeros. The matrix $C_1\ C_2\ \cdot\ \cdot\ \cdot\ C_{n_1+1}$ is obtained from $C_1 C_2\ \cdot\ \cdot\ \cdot\ C_{n_1}$ by inserting the elements $k_{n_1+4,1}$, $k_{n_1+4,2}$,...,k_{n_1+4,n_1+3} in the positions $(n_1+4,1)$, $(n_1+4,2)$, ... , (n_1+4,n_1+3) .

If we assume that $a_{i,i+3} \neq 0$ for all $i=n_1+1,n_1+2,\ldots,$ a continuation of the above procedure leads to the result