ON SOME OPERATORS BETWEEN $\mathcal{L}_P$ - SPACES, $1 \leq P \leq \infty$

THESIS

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INTRODUCTION

INTRODUCTION

Banach space theory, as most abstract mathematical theories, started with the study of particular examples. The Hilbert spaces and the spaces C(K) of continuous functions were the first examples of function spaces which were investigated from the point of view which is now called Banach space theory. The next examples to be studied were the L_p-spaces, $1 \le p \le \infty$.

The appearance of Banach's book [36] signified the beginning of the widespread research and interest in the general theory of normed linear spaces. Banach's book is devoted mainly to the study of what is now called the geometry of Banach spaces.

Since the appearance of Banach's book a huge amount of research was done on the structure of general Banach spaces, as well as the structure of special spaces and in particular of those related to the C(K) and L_p-spaces.

The aim of this thesis is to study $\mathcal{L}_p$-spaces for $1 \le p \le \infty$ and some operator ideals on them.

As main reference about $\mathcal{L}_p$-theory we mention "Classical Banach spaces" of Lindensstrauss and Tzafriri. For the operator ideals reference book we mention "Operator ideals" of A. Pietsch.

The thesis consists of four chapters.

Chapter 0 contains the preliminaries, this chapter is devoted to summarize well known definitions and explain terminology used throughout this thesis. Furthermore various results which are employed without proofs are given.

Chapter I is devoted to studying $\mathcal{L}_p$-spaces. It cannot be considered as Banach functions or sequence spaces. They are spaces whose global structure is quite complicated. It has found that the local point of view is very useful in studying many properties of Banach spaces. The $\mathcal{L}_p$-spaces seem to form a suitable framework in which the study of the isomorphic properties of the classical Banach spaces can be carried out. In this chapter, the proof of some propositions concerning projections and the basis in a separable infinite dimensional $\mathcal{L}_p$-spaces, $1 \le p \le \infty$ are given.

Chapter II is devoted to studying the theory of operator ideals [36] and some operator ideals are mentioned such as absolutely (p, q) summing operators between Banach spaces, $1 \le p$ $q < +\infty$, p-nuclear operators, $1 \le p \le \infty$ and operators of type S_p $1 \le p \le \infty$ (Schatten operators) whose approximation numbers are p-summable. In [6] we obtain a new estimation of absolutely $(S, 1)$ summing norm of the littlewood operators between $\mathcal{L}_u$-spaces. At the end of this chapter, the Banach space X for which $N(X, X) = S_1(X, X)$ is studied. It is known [33] that $N(H, H) = S_1(H, H)$ for any Hilbert space H and a characterization of $\mathcal{L}_p$-spaces using P-nuclear operators and approximation numbers are given.

Chapter III is devoted to studying the concept of limit orders which has been introduced by A. Pietsch as a mean to compare and estimate the norms of different operators ideals. The first three sections of this chapter are devoted to studying the famous work of H. Konig [18]. In section (D) another method of estimating the norm of absolutely summing operators has been studied using the local finite dimensional subspace of $\mathcal{L}_p$-spaces. In [8] we estimate the absolutely 2-summing norm of the littlewood operators A_K between $\mathcal{L}_p$-spaces using the limit order notations. In [9] we give an estimation of the approximation numbers and Gelfand numbers of a matrix operator:

$$T = (\tau_{ij})^{n}_{i=1,}\,{}^{n}_{j=1} : \ell^n_r \longrightarrow \ell^n_r$$

as a special case of our result we obtain the result of [33], in case of finite dimensional spaces, where T is considered as a diagonal operator, we apply our result to littlewood operator.

CHAPTER 0

PRELIMINARIES

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PRELIMINARIES

In this introductory chapter we summarize well known definitions and explain terminology used throughout this thesis. Furthermore, we list various results which are employed without proofs.

A. Operators on finite dimensional linear spaces.

In this section we summarize some basic results from linear algebra which are the back-ground and model for all of the following considerations. Proofs may be found, for example, in the textbook of P. R. Halmos [14].

All linear spaces are considered over the complex field.

Finite dimensional linear spaces:

We denote by $\ell^{(n)}$ the n-dimensional linear space of all complex -valued vectors $x = (\xi_1, \ldots, \xi_n)$.

The spaces ℓ_u^n and ℓ_1^n are the space $\ell^{(n)}$ equipped with the norms

$$\|x\|_u = \left(\sum_{i=1}^{n} |\xi_i|^u \right)^{1/u}$$

and

$$\|x\|_1 = \max_i |\xi_i|$$

respectively.

In case of finite dimensional normed space $(X, \|\ \|)$, we can make an isomorphism between ℓ_u^n and X as follows:

Let $\{e_1, \dots, e_n\}$ be any basis of X. Then every element $x \in X$ admits a unique representation

$$x = \sum_{i=1}^{n} \xi_i e_i \qquad \text{with } \xi_1, \dots, \xi_n \in \mathbb{C}$$

and

$$U : (\xi_i) \longrightarrow \sum_{i=1}^{n} \xi_i e_i$$

defines an isomorphism between ℓ_u^n and X.

Now we formulate the following lemma consulted by [26]

Lemma (1):

Let $(X, \|\cdot\|)$ be finite dimensional normed space with basis $e_1, \dots, e_n$, hence every $x \in X$ can be written as:

$$x = \sum_{i=1}^{n} \lambda_i e_i$$

and we get:

1) $\sum_{i=1}^{n} |\lambda_i|$ defines a norm $\|\cdot\|_1$ on X.

2) There exist constants c_1, c_2 s.t. $c_1 \|x\|_1 \le \|x\| \le c_2 \|x\|_1$

This shows that x runs through a bounded subset iff all (λ_i) are bounded.

Operators and matrices:

By an operator T on a finite dimensional linear space X we always mean a linear map from X into itself.

Let $M=(\mu_{ij})$ be any (n,n)-matrix. Then an operator $\tilde{M}: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ can be defined by:

$$\tilde{M}(\eta)_i = \sum_{j=1}^{n} \mu_{ij}\, \eta_j ,$$

where $\eta = \{\eta_i\} \in \mathbb{C}^n$.

In this way we get a one to one correspondence between (n, n) matrices and operators on $\mathbb{C}^{(n)}$.

On the other hand, let T be an operator on a finite dimensional linear space X. Then, for every fixed basis $z_1 \ldots z_n$ there exists a so-called representing matrix $M = (\mu_{ij})$ determined by

$$T z_j = \sum_{i=1}^{n} \mu_{ij}\, z_i \qquad \text{for } j = 1 \ \ldots \ n$$

With the operator $\tilde{M}$ induced by this matrix, we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & X \\ {\scriptstyle U^{-1}}\downarrow & & \uparrow{\scriptstyle U} \\ \mathbb{C}^n & \xrightarrow{\ \tilde{M}\ } & \mathbb{C}^n \end{array}$$

where U is the isomorphism defined above.

The representing matrix M of a given operator T depends on the underlying basis $z_1 \ldots z_n$. If N is the representing matrix with respect to another basis [illegible] $\ldots$ [illegible] then there exists an invertible matrix A such that $N = A^{-1}MA$. The matrix $A =$ [illegible] is determined by:

$$[illegible] = \sum_{[illegible]}^{n} [illegible]$$

Note that the identity operator denoted by $I_{[illegible]}$ or I is always represented by the unit matrix I_n = [illegible].

Traces:

The trace of an $n \times n$ matrix M = [illegible] is defined by

$$\text{trace } M = \sum_{i=1}^{n} [illegible]$$

Remark (1): [36]

For a matrix M=([illegible]) considering the operator $\tilde{M}: \ell_{[illegible]}^{n} \longrightarrow \ell_{[illegible]}^{n}$ its norm is:

$$\|\tilde{M}\| = \sup \sum_{[illegible]}^{n} [illegible]$$

B. Operators on quasi-Banach spaces:

We use [4] and [51] as standard references.

In the following, by a quasi-norm [illegible] defined on a linear space X we mean a real valued function with the following properties:

1. Let $x \in X$ then $\|x\| = 0$ iff $x = 0$.

2. $\|x + y\| \le K (\|x\| + \|y\|)$ for $x, y \in X$ where $K \ge 1$ is a constant.

3. [illegible] for [illegible] and [illegible]. If $K = 1$ then [illegible] is said to be a norm. In this case condition 2 passes into the well known triangle inequality.

Every quasi-norm $\|.\|$ given on a linear space X induces a metrizable topology. A topology τ on a set X is said to be metrizable if there is a metric d on X which is compatible with τ i.e. the convergence in this metric is equivalent to the convergence in this topology.

A quasi-Banach space is a linear space X equipped with a quasi-norm $\|.\|$ which becomes complete with respect to the associated metrizable topology. This means that all Cauchy sequences are convergent.

In the most important case when $\|.\|$ is a norm, we call X a Banach space.

Theorem (1): *Banach theorem:*

Let X be a Banach space and Y be a normed space. If $T \in L(X,Y)$ is a linear continuous operator and T^{-1} exists then T^{-1} is a linear continuous operator.

Definition (1):-

A mapping $T \in L(X,Y)$ is called an open mapping if it translates each open set in X into an open set in Y.

Remarks (2):

1. In the Banach theorem, continuity of the operator T^{-1} (if it exists) is equivalent to the fact that T is open.

2. Also Banach theorem is a consequence of the open mapping theorem which states that: If X is a Banach space and Y is a