

*On the limit theorems
for
homogeneous Markov chains*

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3- Stochastic processes	2hour / week
4- Theory of reliability	2hour / week
5- Order statistics	2hour / week

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INTRODUCTION

The theory of stochastic processes is generally defined as the "dynamic" part of probability theory, in which one studies a collection of random variables (called a stochastic process) from the point of view of their interdependence and limiting behaviour. The importance of the theory and applications of stochastic processes in scientific disciplines is well established.

Markov chains (processes) is one of the important and major topics of the theory of stochastic processes. It carries the name of the great Russian mathematician A.A.Markov (1856-1922) who formulated the basic definitions and studied the fundamental properties of these chains in the case when the set of possible states is finite.

Up to date Markov chains (processes) are well studied as a part of the theory of general stochastic processes and it has a wide variety of applications in the various scientific disciplines such as Radiotechnology, Psychology, Economics, Epidemiology, Demography and etc... .The application field of Markov chains is related in its extension to the advances

of the Markov chain theory itself, and this, in fact, develops principally in two directions. The first is concerned with the problems of performing the basic classical results of the theory of probability (in particular, limit theorems) of the independence scheme to the case of Markov chains.

The second direction deals with the questions related to the structures of transition probabilities and the analysis of the reality behaviour.

Limit theorems represent, in connection with its theoretical value, the most essential part of modern probability theory, and have, in connection with its practical applications, an exceptional importance.

In fact, the exact formulas and its applications for numerical calculations represent, in probability theory, no more than an exceptional case. This situation generated the necessity of using suitable approximate probability distributions or their related characteristics.

These approximations must, from one side, be useful for numerical calculations and must, from the other side, have

potential ability that ensures the accuracy of the desired approximation. The main aim of limit theorems is to supply approximations that satisfy the indicated requirements.

Limit theorems for homogeneous Markov chains occupy a great distinguished position between all others.

Studies of limit theorems for sequences of the distribution of the sum of random variables, related by a homogeneous Markov chain, are initiated by the work of Markov [15], Romanovski [23], Sarimsakov [24], and Sirazhdinov [25, 26] when the set of possible states of the considered Markov chain is either finite or presented by an interval on the real line.

Latter efforts are directed to extend the achieved results to the case when the set of states of the chain is arbitrary. It is also directed to generalize the well known results such as the central limit theorem, the Berry-Essen inequality, etc...in the case of sequences of independent random variables to the Markov chains case. The fundamental work in these directions is due to Nagaev [16, 17], Skvorcov [30, 31], Sirazhdinov and Formanov [27, 29], and others (see [28]).

The present thesis is devoted to throw some light on the problem of the limit theorems for the distribution of the sum of random variables related by a homogeneous Markov chain. In this connection we are interested in the uniform estimates for the rate of convergence, in these theorems. These estimates will be obtained as a result of studying the structure of the remainder term in these limit theorems.

The thesis consists of three chapters and an appendix. Chapter I, deals with the problem of the uniform estimates for the rate of convergence in the limit theorems for the distribution of the sum of random variables related by a homogeneous Markov chain with arbitrary set of states. Section 1, is a full discussion of the central limit theorem proved in [16], while section 2, is devoted for discussing the uniform estimates for the rate of convergence in the central limit theorem proved in section 1, where the Berry-Essen inequality proved in [17] is discussed, and finally section 3, gives a full exposition of the results of Gharib [9] that generalize the results of sections 1 and 2 to the case when the assumption that the transition probability function has a finite third absolute moment is not fulfilled. In chapter II, the limit theorems for the distribution of the weighted sum of random variables related by a homogeneous Markov chain with arbitrary set of possible states are discussed. Section 1, is devoted to the basic definitions, properties and auxiliary results. In section 2, we present the central limit theorem and the Berry-Essen inequality. The results of this

section are due to Gharib[9]. In section 3, we present our new results concerning the limit theorems in the case when the conditions of the existence of absolute moments of the transition probability function are not fulfilled. In fact, these results generalize the results of section 2, and also, generalize the corresponding results in chapter I, of the unweighted sum of random variables.

In chapter III the problem of the limit theorems for a special type of finite homogeneous Markov chains is discussed. The sequence of random variables related by this type of finite homogeneous Markov chains is called the Markov - Bernoulli sequence of random variables (or the Markov-Bernoulli model) and is introduced by Edwards[4]. The importance of the Markov Bernoulli model comes from its wide variety of practical applications in many fields such as , Reliability, Ecology, etc... (see[18],[21],[32]and[33]). Thus, section 1, introduces the basic definitions and characteristics of the Markov-Bernoulli model. The contents of this section are based essentially on the work of Wang [34] and Anis and Gharib[1]. Section 2, gives a full exposition of the results of Gharib and Yehia [10] concerning uniform estimates of the rate of convergence in the central limit theorem for the Markov-Bernoulli sequence of random variables. In section 3, we present our results concerning the limit theorems in the L_π - space ($1 \leq \pi \leq \infty$) (the space of functions f for which

the norm $\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}}$ is finite) for the Markov-Bernoulli sequence of random variables. The results of this section are new.

The appendix is devoted to some material from functional analysis necessary for the contents of chapters I and II. Specially, the properties and the spectral analysis of the linear operator generated by the transition probability function of the homogeneous Markov chain.

The technique of the proofs of the considered limit theorems in the present thesis, is based on the method of characteristic functions, in consistent with the spectral theory of linear operators in the Banach space of all bounded complex-valued functions.

It is well known that, the method of characteristic functions, represent, in the theory of sums of independent random variables, an analytic and powerful device, which gives the possibility, not only for establishing limit theorems of quality character, but also for proving theorems having expressions of qualitative character, that represent estimates for the rate of convergence in the central limit theorem (the Berry-Essen inequality).

The value of this method, is due further, to the fact that the theory of sums of random variables became an independent branch in the advanced stage of modern probability theory.

CHAPTER I

LIMIT THEOREMS FOR HOMOGENEOUS MARKOV CHAINS WITH ARBITRARY SET OF STATES