FRAGMENTED STRUCTURES OVER FILTERED RINGS
AND
APPLICATIONS TO VALUATION THEORY

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بسم الله الرحمن الرحيم

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SUMMARY

SUMMARY

The study of subrings of a given ring is one of the least developed areas of algebra. A good reason for this is that most properties a ring can have are lost when passing to a subring. The same is certainly true for overrings of a given ring but here there does exist a considerable body of theory because several good examples of overrings that are still related to the original ring do exist, e.g. polynomial extensions, localizations, integral closures, matrix rings, iterated Ore extensions, etc ... . In all these cases the relations between the categories of modules over the rings involved, or subcategories thereof, are stringent enough in order to obtain relations between their structure theories. Of course, classical fields of mathematics like algebraic geometry or number theory are build upon the study of suitable overrings, e.g. local rings containing a coordinate ring, number rings as integral extensions of the ring of integers, etc ... . Perhaps it is an interesting question of almost philosophical nature to ask whether there is more to this than the fundamental fact that people like to deal with the smaller object first before they go on to study the "extensions" of objects or ideas.

In the present work, our main goal is to study an inclusion of rings $S \subset R$ giving both equal rights, i.e. it could well be that R is a well-known ring and we want to describe the structure of S or vice-versa. Even when S and R are closely related properties may travel from S to R much easier than from R to S; in fact it is usually the nature of the relation between S and R that

forces this unilateral nice behaviour. For example if S is a discrete valuation ring and R its field of fractions then S determines R but there are many possible such S in given R.

It is clear that the module categories S-mod and R-mod may not contain too much information about the structural relations between S and R. For example if R is a Galois extension of the field S then Galois theory is not formulated in terms of vector spaces over R but in terms of subfields of R and certain elements in $End_s R$ (e.g. automorphisms fixing S).

Our point of view in this work is that R and S are related by a positive filtration, $S = F_0R \subset F_1R \subset \dots \subset F_nR \subset \dots \subset R$, such that $\cup_n F_nR = R$. The assumption of the existence of such a filtration is a very weak one, however, it is useful because the properties of it are largely controlled by its associated graded ring $G(R) = \oplus_{n\geq 0} F_nR/F_{n-1}R$, where $F_{-1}R = 0$. Note that S is also the part of degree zero of G(R). So we have another important extension $S \subset G(R)$. In order to relate the structure of S and R we introduce the notion of an FR-fragment and on the other hand, FR-fragments are S-modules. We may think of an FR-fragment M as an S-module equiped with "levels" $M = M_0 \supset M_1 \supset \dots \supset M_n \supset \dots$ such that "partial" multiplication with elements of R is possible in the sense that $F_iR\, M_i \subset M$ is well-defined, for all $i \in \mathbb{N}$. The category of FR- fragments is situated between R-mod and S-mod and it has properties resembling those of the category of filtered modules, in particular it is not a Grothendieck category not even an abelian category.

The thesis is divided into four chapters:

The first chapter provides the preliminaries and some background results to be used in subsequent chapters.

In chapter (II) we provide properties of fragments and fragmented morphisms that are of category-theoretical nature, e.g. direct sums, products, tensor product, direct and inverse limits, strict morphisms and exact sequences, ...

Many examples of fragments are given along the way. These examples open the way for several applications, e.g.: glider representations of (finite groups) relating representations of groups and chains of subgroups; relations between modules on different opens in a covering of a non-commutative projective scheme Proj; differential operators and pseudo-differential operators on a ring preferably the coordinate ring of a smooth algebraic variety; relations between modules over the ring of quantum sections of a filtered ring and Rees modules over the Rees ring of the latter, ... Each of these examples will probably lead to an interesting theory but we do not go deeper into these topics here.

On the other hand, one of the most basic examples of fragments is obtained by looking at (non-commutative) valuations and we do expand the related theory for fragments in chapter (IV).

The fundamental underlying result here is the characterization of completely irreducible fragments (see Theorem

(2.3.5)). In the non-commutative case the theory of fragments embraces the theory of Dubrovin valuations in central simple algebras as well as the more general theory of primes in algebras and arithmetical pseudo-valuation, [6], [7] and [30].

The main results of chapter (II) are included in [22].

The examples given provide an exciting outlook on possible applications but first we do have to establish a general theory of- and specific tools for-dealing with fragments. This is our main aim in chapter (III). A large part of this consists in studying the behaviour of fragments under change of rings and the obvious changes one would like to be able to control are localization and integral closure or similar operations. We study the behaviour under localization in the category of S-modules but next to that we pay attention to an intrinsically fragmented localization theory. Paving the way for the envisaged applications in the geometry of quantum algebras we also consider quantum sections and the behaviour of fragments in relation with these.

General fragments have the drawback that they may be fragments with respect to different filtrations FR and moreover the same S- module M may have different chains making it into a different fragment each time. On one hand, most of these ambiguities vanish if we restrict attention to so-called natural fragments, that is to say fragments contained in an R-module and equipped with the "induced" fragment structure, see chapter (II), section 2.1.

On the other hand, an FR-fragment M gives rise to a subring $S \subset S^{(*)} \subset R$, called the ring of definition. The fragment may then be thought of as a tool in relating R and $S^{(*)}$ rather than R and S. In the non-commutative case the filtration defined on $S^{(*)}$ provides an interesting object for study because the different left and right structures involved correspond to positive and negative filtrations. The ring of definition is the instrument relating the theory of fragments to the theory of filtered modules with respect to some specific negative filtration. In specific examples $S^{(*)}$ may be calculated but in general it is not easy task to obtain an explicit description of $S^{(*)}$ in R.

The main results of chapter (III) are included in [23].

Chapter (IV) is devoted to study fragments over commutative and non-commutative valuation rings. We restrict attention to F^{-}-finitely generated fragments (see Definition (4.1.1)). The main results of this chapter are Theorems (4.1.22) and (4.2.4).

The aim of our work is to provide the basic theory of fragments in order to open new possibilities for researching the relations between S and R at least in specific special cases. The examples given motivated the introduction of these concepts and we hope to continue the investigation of the impact of the theory of fragments in several concrete cases, e.g., differential operators, singularities of varieties etc ..., in future work.

CHAPTER 1:

Preliminaries.

CHAPTER I

PRELIMINARIES

In this chapter we recall some basic definitions and elementary results concerning graded and filtered rings as well as localization, orders and various kinds of valuation rings.

Throughout this chapter all rings are associative with units and all modules are unitary left R-modules.

1.1. Filtration on Rings and Modules

Definition 1.1.1. A ring R is said to be a *filtered ring* if there is an ascending chain of additive subgroups of R, say $FR=\{F_nR,\ n\in\mathbb{Z}\}$, satisfying $1\in F_0R$, $F_nR \subseteq F_{n+1}R$ and $F_nRF_mR \subseteq F_{n+m}R$, for all m, $n\in\mathbb{Z}$, [20].

From the definition it is clear that if R is a filtered ring with filtration FR then F_0R is a subring of R.

Definition 1.1.2. Let R be a filtered ring with filtration FR. An R-module M is said to be a *filtered module* if there exists an ascending chain, say $FM = \{F_nM,\ n\in\mathbb{Z}\}$, of additive subgroups of M satisfying: $F_nM \subseteq F_{n+1}M$ and $F_nR\ F_mM \subseteq F_{n+m}M$, for all m, $n\in\mathbb{Z}$. If R and S are filtered rings and M is an R-S-bimodule, denoted $M\in$R-mod-S, then M is said to be a filtered R-S-bimodule if there exists an ascending chain of additive subgroups of M, say $FM = \{F_nM,\ n\in\mathbb{Z}\}$, satisfying: $F_nM \subseteq F_{n+1}M$, $F_nR\ F_mM \subseteq F_{n+m}M$ and $F_mM\ F_nS \subseteq F_{n+m}M$, for all n, $m\in\mathbb{Z}$, [20].

Examples 1.1.3.

(1) An arbitrary ring R may be viewed as a filtered ring if we define the *trivial filtration* FR of R by putting $F_nR=0$ for $n<0$, and $F_nR = R$ for $n \geq 0$. If M is an arbitrary R-module, then any ascending chain of submodules of M defines a filtration and the structure of a filtered R-module on M. The *trivial filtration* on M is defined by $F_{-n}M = 0$, $F_nM = M$ for $n > 0$ and $F_0M = M$, [20].

(2) Let R be a ring and I an ideal of R. The *I-adic filtration* of R is obtained by putting $F_nR = R$, for $n \geq 0$ and $F_nR = I^{-n}$, for $n < 0$. On an R-module M we may then define the *I-adic filtration* of M by putting $F_nM = M$, for $n \geq 0$ and $F_nM = I^{-n}M$, for $n < 0$, [20].

(3) If R is a filtered ring with filtration FR and M is any R-module, then we may define a filtration on M by putting $F_nM = F_nR\ M$, where $F_nR\ M$ is the additive group generated by elements of the form rm with $r \in FR_n$ and $m \in M$. This filtration on M is called the *deduced filtration* (of FR); in this sense the I-adic filtration on an R-module M is the deduced filtration of the I-adic filtration of R, [17].

(4) Let R be a ring and Q an overring of R. Recall that an R-sub-bimodule I of Q is said to be *invertible* (in Q) if there exists another R-sub-bimodule J in Q such that IJ=JI=R, [17].

If I is an ideal of R such that it is invertible then

$$R \subseteq I^{-1} = \{q \in Q:\ qI \subseteq R\}$$
$$= \{q \in Q:\ Iq \subseteq R\}.$$

Hence we have

$$\ldots \subseteq I^2 \subseteq I \subseteq R \subseteq I^{-1} \subseteq \ldots$$

and $R' = \bigcup_{n\in\mathbb{Z}} I^n$ is a filtered ring with filtration FR':

$$F_nR' = \begin{cases} R & \text{if } n = 0 \\ I^{-n} & \text{if } n \neq 0 \end{cases}$$

Definition 1.1.4., [20]. Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM.

(1) If $F_nM = 0$ for $n < 0$, then FM is called a *positive filtration* (in this case we also say that M is positively filtered). If there exists an $n_0 \in \mathbb{Z}$ such that $F M_p = 0$, for all $p < n_0$, then FM is called *discrete.*

(2) If $M = \bigcup_{n\in\mathbb{Z}} F_nM$, then FM is called *exhaustive.*

(3) If $\bigcap_{n\in\mathbb{Z}} F_nM = 0$, then FM is called *separated.*

Let R be a filtered ring with filtration FR and let M, N be filtered R-modules with filtrations FM, FN, respectively. Then $f \in Hom_R(M,N)$ is said to be of degree ω if $f(F_nM) \subset F_{n+\omega}N$, for all $n \in \mathbb{Z}$. The set of all homomorphisms of finite degree, denoted by $HoM_R(M,N)$, is a subgroup of $Hom_R(M,N)$ with a natural filtration $F(HoM_R(M,N))$, where

$$F_n(HoM_R(M,N)) = \{f \in HoM_R(M,N):\ f \text{ is of degree } n\}\ ,\ n \in \mathbb{Z}.$$