ON APPROXIMATION OF FUNCTIONS AND OPERATORS

Thesis
Submitted for the Degree of
DOCTOR OF PHILOSOPHY
In Science (Pure Mathematics)

Presented By

Amal Talaat Mohamed Matter
B.Sc. In Mathematics 1983 (Ain-Shams University)
M. sc. In Mathematics 1987 (Ain-Shams University)

Supervised By

Prof. Dr. Entisarat El-Shobaky
Professor of Pure Mathematics Faculty of science Ain-Shams University

Dr. Adel A. El-Kady
Lecturer of Pure Mathematics
Faculty of Science
Ain-Shams University

Dr. Labib R. El-Sayed
Lecturer of Pure Mathematics
Faculty of Science
Ain-Shams University

Submitted To

Faculty of Science
Ain-Shams University
Cairo Egypt
1993

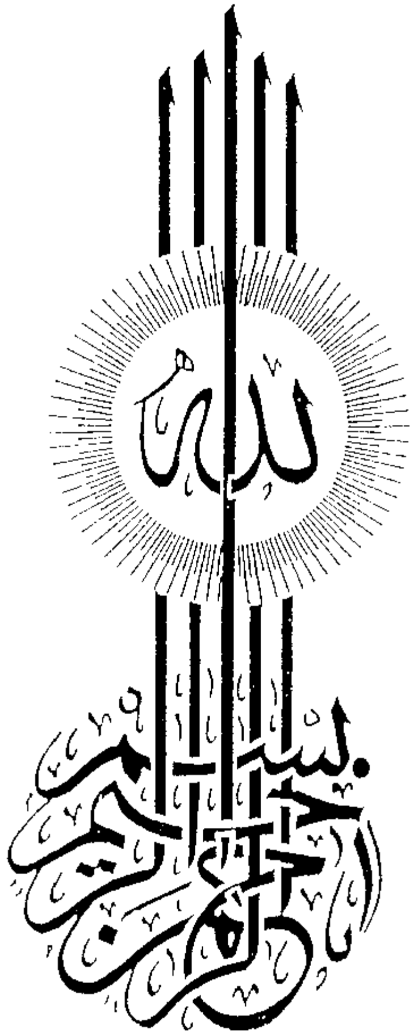
ACKNOWLEDGMENT

I would like to express my deepest gratitude and thanks to *Prof. Dr. Entisarat El-Shobaky*, Professor of Pure Mathematics, Faculty of Science, Ain Shams University, for suggesting the point of research, her continuous supervision, valuable suggestions, and keen interest.

My thanks to *Dr. Adel A.El-Kady*, Lecturer of Pure Mathematics, Faculty of Science, Ain Shams University, and *Dr.Labib R. El-Sayed* Lecturer of Pure Mathematics, Faculty of Science, Ain Shams University, for their help and interest.

Thanks to the head and the members of the Mathematics Department Faculty of Science Ain-Shams University for their encouragement and providing the necessary facilities.

Finally I would like to express my deepest thanks to my husband, *Dr. S. Y. Elzaiat*, Lecturer of Physics, Faculty of Science, Ain Shams University for his continuous encouragement and typing the thesis.


بسم الله الرحمن الرحيم

Contents

INTRODUCTION AND SUMMARY

INTRODUCTION

The year 1885 was an important year for approximation theory. In that year, *Weierstrass* and *Runge* announced well-known approximation theorems bearing their names. Since then, several important extensions of the theorems have been obtained by *De La Vallée Poussin* [1], *Bernstein* [2], *Stone* [3], and others [4].

The well-known theorem of *Stone*, referred to in the literatures as the *Stone-Weierstrass* theorem, characterizes dense subalgebras, A, of the algebra C(X) of continuous real valued functions on a compact set X in the uniform topology in terms of separation properties of points in X by functions in A.

Chapter zero contains all basic definitions which have been used through the thesis.

Chapter one is an introductory chapter on the approximation theory. This chapter is devoted to the treatment of standerd linear problems, non-linear problems and in particular the case of rational approximations.

Although the *Lebesgue* spaces L_p $(1 \le p \le \infty)$ play a primary role in many areas of Mathematical analysis. These L_p spaces are insufficient to describe the mapping properties of operators such as the conjugate- function operator [1-3].

There are classes of *Banach* spaces of measurable functions that are also of interest. The larger class of these spaces is known as *Orlicz* spaces, which are direct generalization of the spaces L_p, and of intrinsic importance. The *Orlicz* spaces involve functionals of the form $\int\varphi(|f|)$ for increasing functions φ. These spaces are investigated in some detail in [5-8].

In chapter two, we begin by studying *uniform* integrabilty principle and then a study on convex functions and Young functions are given. Also the *Orlicz* spaces and its properties are presented. Then our extension of periodic functions expansion in that spaces is given.

It is known that, to every continuous periodic function $f(x)$ and every positive integer n, there corresponds a completely defined trigonometric sum denoted by $T_{n-1}(x,f)$ which deviates the least from $f(x)$. The measure of the deviation:

$$c_n = \max_{0\le x\le 2\pi} |f(x)-T_{n-1}(x,f)|$$

is a functional on $f(x)$ dependent on n.

One can generalize the above statment to operators on *Banach* spaces in the following manner. Let E and F be two *Banach* spaces. For each mapping $T\in L(E,F)$, (where $L(E,F)$ is the space of all bounded linear operators from E to F), there exist the approximation numbers, $a_n(T)$ with $n=0,1,2,\ldots,$ provide a measure of how well T can be approximated by finite mappings whose range is at most n-dimensional.

Chapter three, is devoted to the structure and analysis of classes of linear operators between Orlicz spaces and some applications. We first consider factorization properties as well as characterizations of linear integral operators on Orlicz spaces. Then we follow some famous works of König and others to give generalization of their works to Orlicz spaces. Specially we give estimates of approximation numbers of integral operators between Orlicz spaces. Also we approximate some classes of absolutely summing operators by finite operators. Finally, we study the case of approximations of $K_n(X,c_o)$ in $L(X,c_o)$. Then an application of this work in parallel computation is given.

REFERENCES

[1] Vallée Poussin de La; Sur lápproximatuis des functions, d'une variable réele et de leurs dérivées par des polynomes et des suites finies de Fourier', *Bull. Acad. Sci. Belgique* (1908) 193-254.

[2] S. N Bernstien, Lecons sur les properiétés extermales et la meilleure approximation des fonctions analytiques d'une variable reele , Gauthier-Villars Paris 1926.

[3] M. H. Stone, 'Applications of the theory of Boolean rings to general topology' *Trans. Amer. Math. Soc.* 41 (1937) 375-381.

[4] W. Orlicz, 'Über eine gewisse Klasse von Raümen vom Typus B'', Bull. Intern. de l'Acad. Pol. serie A, Cracavie, 8 (1932) 207-220.

[5] W. Orlicz, 'Über Räume (L^M)'', Bull. Intern. de l'Acad. Pol. serie A, Cracavie (1936), 93-107.

[6] W. A. J. Luxemburg and A. C. Zaanen, "Conjugate spaces of Orlicz spaces", Nederl. Akad. Wetensch. Prac. ser. A59-Indag. Math. 18(1956).

[7] A. C. Zaanen, "Linear Analysis", New York and Amsterdam, (1953).

[8] J. Lindenstrauss, L. Tzafriri, "Classical Banach Spaces I,II", Springer-Verlag, Berlin Heidelberg New York, (1977), (1979).

CHAPTER 0

BASIC DEFINITIONS AND SOME RESULTS

CHAPTER 0

BASIC DEFINITIONS AND SOME RESULTS

Introduction:

In this chapter we present all basic definitions which will be used in the following chapters. Also we introduce some propositions and theorems which should be known before studying the approximation of functions and operators.

0.1- Weierstrass's Polynomial Approximation Theorem [33]:

Let $f(x)$ be a real valued or complex-valued continuous function on the closed interval $[0,1]$. Then there exists a sequence of polynomials $P_n(x)$ which converges as $n \to \infty$ to $f(x)$ uniformly on $[0,1]$. According to S. Bernstein, we may take:

$$P_n(x) = \sum_{p=0}^{n} C_p^n f\left(\frac{p}{n}\right) x^p (1-x)^{n-p} .$$

0.2- Stone-Weierstrass Theorem [33]:

Let X be a compact space and $C(X)$ the totality of real-valued continuous functions on X. Let a subset B of $C(X)$ satisfy the three conditions:

(i) If $f, g \in B$, then the function product $f.g$ and linear combinations $\alpha f + \beta g$, with real coefficients α, β, belong to B,

(ii) the constant function 1 belongs to B, and:

(iii) the uniform limit f_∞ of any sequence $\{f_n\}$ of functions belong to B also belongs to B. Then $B = C(X)$, iff B separates the points of X, i.e iff for every pair (s_1, s_2) of distinct points of X, there exists a function x in B which satisfies

$$x(s_1) \neq x(s_2).$$

0.3- Definition [18]:

A real valued function $M(x)$ of a real variable x is called convex if the inequality;

$$M\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\,[M(x_1) + M(x_2)]$$

holds for all values of x_1, x_2.

0.4- Jensen's Inequality [8]:

Let (X, S, μ) be a measure space with $\mu(X)=1$, if ϕ is convex on (a, b) where $-\infty<a<b<\infty$ and f is an integrable function such that $a < f(x) < b$ for all x, then;

$$\phi\left(\int f\, d\mu\right) \leq \int \phi\circ f\, d\mu.$$

Note:

If $X = [x_1, x_2, \ldots, x_n]$, $S = P(x)$, $\mu([x_i]) = \alpha_i \geq 0$.

Then Jensen's inequality is;

$$\phi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \phi(x_i),$$

where $\sum_{i=1}^{n} \alpha_i = 1$.

0.5-Lemma:

Let f be a twice continuously differentiable real-valued function on an open interval (a, b). Then, f, is convex iff its second derivative, f'', is non-negative throughout (a, b).

Here are some functions on, $\mathbb{R}$, whose convexity is a consequence of above lemma.

1) $f(x) = e^{\alpha x}$, where $-\infty<\alpha<\infty$.

2) $f(x) = x^p$ if $x\geq 0$, $f(x) = \infty$ if $x<0$, where $1\leq p\leq\infty$.

3) $f(x) = -x^p$ if $x\geq 0$, $f(x) = \infty$ if $x<0$, where $0\leq p\leq 1$.

4) $f(x) = x^p$ if $x>0$, $f(x) = \infty$ if $x\leq 0$, where $-\infty<p\leq 0$.

5) $f(x) = (\alpha^2-x^2)^{-1/2}$ if $|x|<\alpha$, $f(x)=\infty$ if $|x|\geq\alpha$, where $\alpha>0$.

6) $f(x) = -\log x$ if $x>0$, $f(x)=\infty$ if $x\leq 0$.

7) $f(x) = x\log x$ is convex on $(0,1)$.

8) $f(x) = \dfrac{|x|^\alpha}{\alpha}$, $\alpha>1$.

9) $f(x) = (1 + |x|)\ln(1 + |x|) - |x|$.

10) $f(x)=e^{|x|} - |x| - 1$

0.6- Definition [12]:

Let X,Y be vector spaces. An operator $T:X\longrightarrow Y$ is linear if

$$T(ax+by) = aTx + bTy,$$

for all scalars a,b and all $x,y\in X$.

0.7-Definition [12]:

Let X, Y be normed spaces. An operator $T : X \longrightarrow Y$ is bounded if there is a number m such that:

$$\|Tx\|\leq m\|x\|$$

0.8-Definition:

If X and Y are two normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y.

0.9- Theorem (Banach spaces of operators) [12]:

The space $L(X, Y)$ of bounded linear operators $X \longrightarrow Y$ is a Banach space if Y is Banach space.

0.10- Proposition (Adjoint Operator) [12]:

If $A: H \longrightarrow H$ is a bounded linear operator on a Hilbert space H, then there is a unique operator $A^*: H \longrightarrow H$ such that:

$$\langle x, A^*y \rangle = \langle Ax, y \rangle \qquad \text{for all } x, y \in H.$$

A^* is linear and bounded, $\|A^*\| = \|A\|$, and $(A^*)^* = A$. A^* is called the adjoint of A.

0.11-Definition [12]:

An operator T on a normed space is compact if it maps every bounded set into a relatively compact set.

0.12- Definition [12]:

The rank of a linear operator is the dimension of its range.

0.13- Definition:

If X and Y are two normed linear spaces, then $K(X, Y)$ denotes the set of all compact operators in $L(X, Y)$, and $K_n(X, Y)$ consisting of all operators of rank n.

0.14- <u>Definition</u> [12]:

An operator A on an inner product space is called:

positive-definite	if $(x, Ax) > 0$ for all $x \neq 0$,
negative-definite	if $(x, Ax) < 0$ for all $x \neq 0$,
positive	if $(x, Ax) \geq 0$ for all x,
negative	if $(x, Ax) \leq 0$ for all x,

0.15- <u>Definition</u> [13]:

For every operator $T \in L(X,Y)$ the n-th approximation numbers are defined by :

$$a_n(T)=\inf \{\| T-T_n \| : T_n \in L(X,Y) \text{ and rank } T_n \leq n \}.$$

The basic properties of approximation numbers are set out in the following proposition.

0.16- <u>Proposition</u> [13]:

Let X, Y, Z be Banach spaces and let $f \in L(X,Y)$ and $g \in L(Y,Z)$. Then:

(1) $a_{m+n}(g \circ f) \leq a_m(g) a_n(f)$ for all $m,n \geq 0$.

(2) $a_{m+n}(g+f) \leq a_m(g)+a_n(f)$ for all $m,n \geq 0$.

(3) $a_n(\lambda f) = |\lambda| a_n(f)$ for all $n \geq 0$, where λ is a scalar.

(4) $\|f\| = a_0(f) \geq a_1(f) \geq \dots \geq a_n(f) \geq a_{n+1}(f) \geq \dots \geq 0$, so that $f \in L(X,Y)$ iff $(a_n(f)) \in \ell^\infty$.

(5) If $(a_n(f)) \in c_0$, then $f \in K(X, Y)$, where c_0 is the linear space of all sequences that converge to zero.

(6) $a_n(f) = 0$ iff $f \in T_n(X,Y)$, where $T_n(X, Y)$ is the subspace of $L(X, Y)$ of all operators with rank at most n.

(7) $a_n(f') \leq a_n(f)$ for all n.