ON GENERALIZATIONS OF CONTINUOUS MODULES

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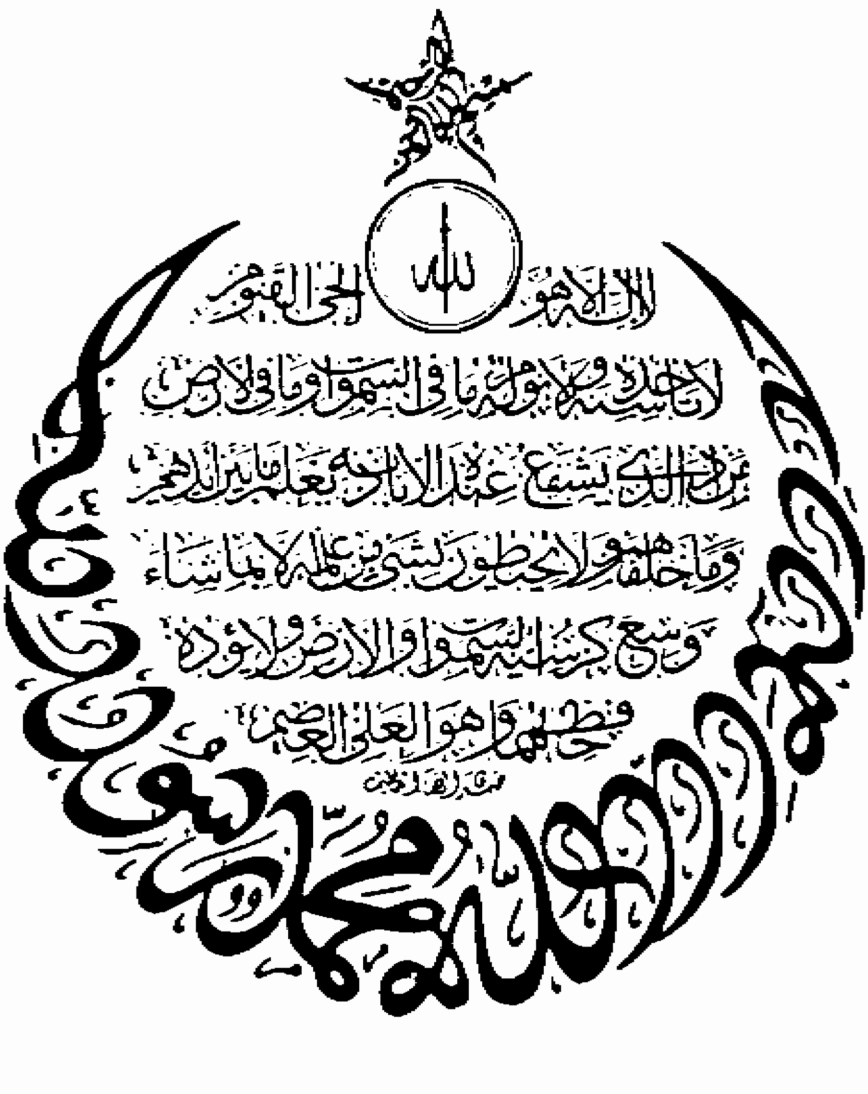
الله
لا إله إلا هو الحي القيوم
لا تأخذه سنة ولا نوم له ما في السموات وما في الأرض
من ذا الذي يشفع عنده إلا بإذنه يعلم ما بين أيديهم
وما خلفهم ولا يحيطون بشيء من علمه إلا بما شاء
وسع كرسيه السموات والأرض ولا يؤوده
حفظهما وهو العلي العظيم
صدق الله العظيم
لا إله إلا الله محمد رسول الله

بسم الله الرحمن الرحيم

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M.Sc. COURSES

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	hours per week
1. Homology Theory	2
2. Functional Analysis	2
3. Modern Algebra	2
4. Differential Equations	2
5. Numerical Analysis	2

CONTENTS

SUMMARY

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Injective and quasi-injective modules play an important role in module theory, and (quasi-) continuous modules are a generalization of these concepts. One of the interesting results, on the study of (quasi-) continuity, is that many of the important properties that hold for (quasi-) injective modules, still hold for (quasi-) continuous modules. It is often more convenient to work with (quasi-) continuity rather than the notion of (quasi-) injectivity.

The origin of the concept of continuous modules lies in Von Neumann's continuous geometries [39], where the continuous regular rings are the coordinate rings of these geometries.

Y. Utumi (1960) studied continuous regular rings extensively, and provided many of their characterizations in lattice theoretical terms [37]. A general definition of continuity for arbitrary rings was given, again by Utumi in 1965 [38]:

A ring R is called a right continuous if (i) every right ideal of R is essential in a direct summand of R, (ii) if a right ideal I is isomorphic to a summand of R, then I is a summand of R. Note that every regular ring automatically satisfies condition (ii) (see Lemma (2.2.5)). He also utilized the condition (iii): for any two right ideals I_1, I_2, with $I_1 \cap I_2 = 0$, the projection $I_1 \oplus I_2 \twoheadrightarrow I_j$, $j=1,2$, is given by left multiplication by a ring element.

The concept of continuity and quasi-continuity were generalized to modules by L. Jeremy [17], and S. Mohamed and T. Bouhy [25]. They, independently, applied Utumi's conditions (i), (ii) to modules, and obtained what they called the conditions (c_1) and (c_2).

The endomorphism ring of a quasi-continuous module was studied by L. Jeremy [17].

S. Mohamed and T. Bouhy [25], studied continuous modules and showed that every (quasi-) injective module is continuous. The converse is not true (example (2.1.5)).

The condition (iii) was extended to modules by V. Goel and S. Jain [11], and was called π-injectivity. It is equivalent to quasi-continuity, and also to the finite extending property defined by M. Harada [15].

B. Müller and T. Rizvi ([28], [29], [30], [31]) dealt with several important aspects of the theory of (quasi-) continuous modules, analogously to the concept of injective hulls defined by Eckmann [6]. They introduced the concept of continuous hull, and explicitly described the continuous hulls for the class of uniform cyclic modules, and of non-singular cyclic modules over commutative rings.

The characterization of direct sums of quasi-continuous modules, in full generality, due to B. Müller and T. Rizvi [31], was demonstrated independently by C. Muck [27]. The (easy) special case was observed in [17] and in [11].

A. W. Chatters and C. R. Hajaranavis [5] investigated rings with chain conditions in which every complement right ideal is a direct summand.

The condition (c_1) was studied by M. Kamal [19], and by M. Kamal and B. Müller ([20], [21], [22]).

Several variations of that condition were investigated, in numerous papers by M. Harada [15] and his collaborators, under the heading of "extending properties".

The present thesis, which consists of three chapters, focuses on some important aspects of the theory of continuous and quasi continuous modules.

The first chapter provides the preliminaries and some background results to be used in subsequent chapters.

In the second chapter we state the defining conditions, and provide the basic results, of continuous modules. Such defining conditions hold for quasi-injective modules. In section 2, we study the structure of the endomorphism rings of continuous modules. As a result, the endomorphism rings of continuous modules possesses the most crucial properties of the endomorphism rings of quasi-injective modules. The endomorphism rings of injective modules were first studied by Y. Utumi [37], and latter generalized to quasi-injective modules by C. Faith and Y. Utumi [8]. S. Mohamed and T. Bouhy [25] generalized these results to continuous modules. In section 3, we give the decomposition theorem of continuous modules into direct sums of square free and quasi-injective submodules. As a result, in conjunction with some other theorems, in section 4, we show that continuous modules have the exchange property. The finite exchange property for continuous modules was obtain from Warfield [41]. Theorem (2.4.7) is due to Mohamed and Müller [26]. Beyond these facts, we discuss in sections 5 and 6, continuous modules over commutative Noetherian rings and provide some classes of rings for which every continuous module is quasi-injective.

The third chapter deals with quasi-continuous modules. In the first section we study the injective hulls of quasi-continuous modules. As a result, we determine all quasi-continuous modules which are essential in injective modules. In section 2, we investigate the relation between the concepts of relative injectivity and of quasi-continuity. We provide the concept of π-injectivity, which is a generalization of that of relative injectivity, and show that the concept of π-injectivity is equivalent to that of quasi-continuity. In section 3, we investigate the direct sums of decomposition of quasi-continuous modules, and give the necessary and sufficient conditions for such direct sums to be quasi-continuous. We study in section 4, the decomposition theorem of quasi - continuous modules into direct sums of indecomposable (uniform) submodules. In section 5, we characterize directly finite and purely infinite quasi-continuous modules, and show that every quasi-continuous module is a direct sum of directly finite and purely infinite submodules. Section 6, deals with the endomorphism rings of quasi-continuous modules. We study quasi-continuous modules with some conditions on their endomorphism rings. We end this chapter by giving the structure of quasi-continuous modules over Ore domains, then we give the full characterization of quasi-continuous abelian groups.

Most of the materials of the last chapter are due to B. Müller and T. Rizvi [28], [30], [31], V. Goel and S. Jain [11], L. Jeremy [17], and B. Müller and M. Kamal [20], [21], [22]. Some of the proofs we have given are slightly different, and some others are easier and completely different from the ones originaly given by the above mentioned authers.

The results of Section 7 of the last chapter seem to us to be original.

CHAPTER I

BASIC DEFINITIONS AND RESULTS

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BASIC DEFINITIONS AND RESULTS

In this chapter, we introduce the definitions and the conventions which are required for our study in this thesis. The first section provides the preliminaries and some background results to be used in subsequent chapters. In the second section we provide some well known facts for latter use, concerning the concept of injectivity and some of its generalizations. The third section deals with the exchange and cancellation properties. Finally we concern with the directly finite and purely infinite modules.

Throughout this chapter all rings (denoted by R) are associative with units, and all modules are unitary right R-modules. R need not be commutative except in some special cases and it will be mentioned explicitly.

§.1.1. **Preliminaries**:

Definition (1.1.1):

A submodule K of an R-module M is said to be essential in M (denoted by $K \leq' M$), if $K \cap L \neq 0$, for every non-zero submodule L of M.

In this case we may say that M is an essential extension of K. It is clear that $K \leq' M$ iff for any $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq mr \in K$.

A monomorphism $f: K \longrightarrow M$ is said to be essential if $\operatorname{Im} f \leq' M$.