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THE CLASSIFICATION OF COMPLEX ANALYTIC MAPS ON COMPLEX ANALYTIC MANIFOLDS

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M. SC. COURSES
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(i) Theory of functions of matrices 2 hours per week
(during the whole academic year)

(ii) Numerical linear algebra 2 hours per week

(iii) Abstract algebra 2 hours per week

(iv) Functional analysis 2 hours per week

(v) Partial differential equations 2 hours per week

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P R E F A C E

The purpose of the thesis is to study the classification of the analytic map $M \longrightarrow G/H$, through an initiated analytic map which we call a fully analytic map, with M a complex analytic manifold and G/H the space of all cosets of H in G, G being a complex Lie group and H a closed subgroup of G.

Indeed, we prove the following theorem:

"Let $\bar{\Phi} : T^{(s+1)}(M) \longrightarrow L(G/H)$ be a fully analytic map, then there exists an analytic map $\phi : M \longrightarrow G/H$ such that $\Phi = w \circ d\phi$ on $T(M)$, where w is a vector form and $d\phi$ is a tangent map".

Throughout the dissertation all manifolds, maps, vector bundles, cross sections are supposed to be ~~analytic~~ of class C^w.

The layout of the thesis is as follows.

In Chapter I, we give a short review of some facts which we shall need later on.

In Chapter II, we give the proof of the theorem stated above.

Chapter III , is devoted to some applications of the theorem.

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C O N T E N T S

CHAPTER I

A GENERAL SURVEY ON COMPLEX ANALYTIC MANIFOLDS

In this chapter we give some preliminaries which will be used later on. We mention briefly some properties of, and some structures on, complex analytic manifolds, tangent spaces, Lie groups and Lie algebras.

The chapter is divided into three sections:

1. Complex analytic manifolds, tangent spaces and distributions.
2. Some concepts on the theory of Lie groups and Lie algebras.
3. Theory of fibre bundles.

Throughout this chapter we give some examples to explain the ideas of the definitions and concepts which are basic in our work.

1. COMPLEX ANALYTIC MANIFOLDS, TANGENT SPACES and DISTRIBUTIONS

Throughout this section we follow [1], [2], [3] and [8].

Definition (1.1.1): A complex analytic manifold of dimension n is a Hausdorff space covered by neighbourhoods each of which homeomorphic to a cell in an n-dimensional complex space, C^n (i.e., there is a finite open covering $\{U_\alpha\}$ and foreach α,

$$u_\alpha : U_\alpha \longrightarrow C^n ,$$

is a homeomorphism of U_α onto an open subset of C^n) and such that, for any two neighbourhoods U_α and U_β with non-empty intersection, the map:

$$u_\beta \circ u_\alpha^{-1} : u_\alpha (U_\alpha \cap U_\beta) \longrightarrow u_\beta (U_\alpha \cap U_\beta)$$

is an analytic map.

The above conditions are called a complex analytic structure and the pair (U_α, u_α) is called a local chart.

Definition (1.1.2): The functions defining u_α are called local coordinates in U_α. Also, if ($z^1, \ldots, z^n$) $\in C^n$, then the system of functions $z^1 \circ u_\alpha, z^2 \circ u_\alpha, \ldots, z^n \circ u_\alpha$ defined in U_α is called a local coordinate system in U_α and simply written as $\{ z^i \}_{i=1,\ldots,n}$.

Definition (1.1.3): U_α is defined to be a cubic neighbourhood of a point $p \in M$, if it is possible to find a chart (U_α ; u_α) such that $u_\alpha(p)$ is the origin of C^n defined by:

$$| z^1 | < b, \ldots, | z^n | < b$$

for a positive number b.

Remark (1.1.1): A complex manifold has an even-dimension , for there is a one-to-one correspodence between complex coordinates ($z^1, \ldots, z^n$) and real coordinates ($x^1, \ldots, x^n$, $y^1, \ldots, y^n$), by setting:

$$z^k = x^k + \sqrt{-1}\ \ y^k \ ; \quad k = 1, \ldots, n$$

with $\bar{z}^k = x^k - \sqrt{-1}\ \ y^k$ is the conjugate of z^k.

Example (1.1.1): The n-dimentional complex vector space, C^n, forms a complex analytic manifold. For the identity map I in C^n, together with the chart (C^n, I) forms a complex structure on C^n .

Example (1.1.2): The complex projective space P_n (= space of complex lines through the origin in C^{n+1}) forms a complex manifold where the complex structure is introduced as follows:

Let ($t_o, t_1, \ldots, t_n$) be complex coordinates on P_n, and define:

(i) $U_j = \left\{ p \in P_n \ ; \ t_j \ (p) \neq o \right\}$, for $j = o, 1, 2, \ldots, n$.

(ii) The analytic functions

$u_j : U_j \longrightarrow C^{n+1}$,

$(t_o, t_1, \ldots, t_n) \longrightarrow (Z_{jo}, \ldots, \hat{Z}_{jj}, \ldots; Z_{jn})$

where $Z_{jk} = t_k \ / \ t_j$ for $t_j \neq o$ and by $\hat{}$ we mean that this term is omitted.

Thus (U_j , u_j) forms a chart for all $j = o, 1, \ldots, n$ and then the complex structure is introduced.

Let M be a complex analytic manifold, p be any point of M and let $\mathcal{F}$ (p) denote the algebra of all analytic functions defined in a neighbourhood of p.

Definition (1.1.4): A complex tangent vector X_p at $p \in M$ is a map:

$$X_p : \mathcal{F}(p) \longrightarrow C,$$

such that:

$$X_p (af + bg) = a X_p f + b X_p g ,$$

$$X_p (fg) = f (p) (X_p g) + (X_p f) (g (p))$$

for all $a, b \in C$ and $f, g \in \mathcal{F}$ (p).

All tangent vectors at a point $p \in M$ form a vector space T_p (M) over C.

Definition (1.1.5): The tangent space T (M) is defined to be $\bigcup_{p \in M} T_p (M)$.

Remark (1.1.2): Let $\{Z^i\}_{1 \leq i \leq n}$ be a local coordinate system at $p \in M$, then $\left\{\frac{\partial}{\partial Z^i}\right\}_{1 \leq i \leq n}$ form a basis for $T_p(M)$.

Remark (1.1.3): T_p (M) is a 2n-dimensional real vector space, with $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}_{1 \leq i \leq n}$ considered as a basis.

Let N and M be two analytic manifolds.

Definition (1.1.6): The map $\Phi : N \longrightarrow M$ is said to be analytic at a point $p \in N$, if for each $f \in \mathcal{F}(\Phi(p))$, $f \circ \Phi$ is an analytic function at $p \in N$.

Definition (1.1.7): Let $\Phi : N \longrightarrow M$ be an analytic map at a point $p \in N$, then the differential of Φ is the (tangent) map

$$d\Phi \; : \; T_p(M) \longrightarrow T_{\Phi(p)}(N)$$

provided that:

$$\left\{ d\Phi (X_p) \right\} \quad f = X_p (f \circ \Phi) ,$$

for $\qquad X_p \in T_p(N)$ and $f \in \mathcal{F}_p(M)$.

Proposition (1.1.1): Let M_1, M_2 and M_3 be three analytic manifolds . Considering any two analytic maps:

$$\Phi : \quad M_1 \longrightarrow M_2 ,$$

$$\Psi : \quad M_2 \longrightarrow M_3 ,$$

we have $d(\Psi \circ \Phi) = d\Psi \circ d\Phi$.

Proof:

Let $p \in M_1$, $r \in \Psi(\Phi(p))$ and $h \in \mathcal{F}(r)$.

Then it is clear that

$h \circ (\Psi \circ \Phi) = (h \circ \Psi) \circ \Phi$ in any neighbourhood of p.

By using definition (1.1.7) we have

$$\left[d(\Psi \circ \Phi) \; X_p \right] h = X_p \left[h \circ (\Psi \circ \Phi) \right]$$

$$= X_p \left[(h \circ \psi) \circ \phi \right]$$
$$= d\phi(X_p)\ (h \circ \psi)$$
$$= d\psi \left[d\phi(X_p) \right] h.$$

Thus $d(\psi \circ \phi) = d\psi \circ d\phi$. Q.E.D.

Definition (1.1.8): The analytic map $\phi : N \longrightarrow M$ is said to be regular at $p \in N$ if $d\phi_p$ is an injective mapping.

Definition (1.1.9): The analytic map $\phi : N \longrightarrow M$ is called an analytic diffeomorphism, if ϕ^{-1} is an analytic map.

Definition (1.1.10): A subset N of a manifold M is called a submanifold if:

(1) N is itself an analytic manifold,

(2) The injective map $\phi : N \longrightarrow M$ is regular.

Let q, n be the dimensions of N,M respectively.

Definition(1.1.11): A closed submanifold N of a manifold M is a submanifold such that:

(1) $\phi(N)$ is closed in M where $\phi : N \longrightarrow M$

(2) If $p \in N$, then there is a neighbourhood U of p with local coordinates $(Z^1, \ldots, Z^n)$ such that $N \cap U$ is the set of points in M at which $Z^{q+1}, \ldots, Z^n$ all vanish.

Definition(1.1.12): A vector field X on M is a map,

$$X : M \longrightarrow T(M) : p \longrightarrow X_p, \text{ for each } p \in M.$$

The vector field X is said to be analytic at $p \in M$ if for each $f \in \mathfrak{F}(p)$, Xf is analytic at p, where Xf is defined by $(Xf)_p = X_p f$.

The vector field X can be expressed by

$$X = \sum_i a^i \frac{\partial}{\partial Z^i}$$

with respect to a local coordinate system $\{Z^i\}_{1 \leq i \leq n}$ where the components a^i are functions defined in a neighbourhood of $\{Z^i\}_{1 \leq i \leq n}$.

If a^i's are analytic functions, then the vector field is analytic.

Definition(1.1.13): Let X, Y be two analytic vector fields on M. The commutator, $[X,Y]$, of X and Y is defined by

$[X,Y]_p f = X_p(Yf) - Y_p(Xf)$, $f \in \mathcal{F}(p)$.

The bracket $[X,Y]$ is also an analytic vector field.

For, let

$$X = \sum a^j \frac{\partial}{\partial Z^j}, \quad Y = \sum b^j \frac{\partial}{\partial Z^j} .$$

Then

$$\begin{aligned} [X,Y] f &= X(Yf) - Y(Xf) \\ &= \sum a^j \frac{\partial}{\partial Z^j} \left(\sum b^k \frac{\partial f}{\partial Z^k} \right) - \sum b^j \frac{\partial}{\partial Z^j} \left(\sum a^k \frac{\partial f}{\partial Z^k} \right) \\ &= \sum_{j,k} \left(a^j \frac{\partial b^k}{\partial Z^j} - b^j \frac{\partial a^k}{\partial Z^j} \right) \frac{\partial f}{\partial Z^k} . \end{aligned}$$

Thus

$$[X,Y] = \sum_{j,k} \left(a^j \frac{\partial b^k}{\partial Z^j} - b^j \frac{\partial a^k}{\partial Z^j} \right) \frac{\partial}{\partial Z^k} .$$

Hence $[X,Y]$ is a vector field with components

$$c^{jk} = a^j \frac{\partial b^k}{\partial z^j} - b^j \frac{\partial a^k}{\partial z^j} .$$

Since a^j, b^j are analytic functions, then c^{jk} is analytic and $[X,Y]$ is an analytic vector field.

The commutator has the following properties:

(1) $[aX, bY] = ab[X,Y]$ for $a, b \in C$

(2) $[X,Y] = -[Y,X]$

(3) $[[X,Y], Z] + [[Y,Z], X] + [[Z,X], Y] = 0$,

which is called Jacobi's identity.

Definition (1.1.14): A distribution Δ of dimension r on a complex analytic manifold M is a law which assigns to every point $p \in M$ a subspace $\Delta(p)$ of dimension "r" of $T_p(M)$. Δ is called analytic, if every point $p \in M$ has a neighbourhood U and r-analytic vector fields. $X_1, \ldots, X_r$ on U, such that $X_{1p}, \ldots, X_{rp}$ form a basis for $\Delta(p)$ at $p \in U$.

The set $\{X_1, \ldots, X_r\}$ is called a local basis for the distribution Δ in a neighbourhood of p.

Definition (1.1.15): Δ is called involutive, if whenever X and Y belong to Δ, then so does their Poisson's braket $[X,Y]$.

Definition (1.1.16): An integral manifold of Δ is a connected submanifold N of M such that $\Delta(p)$ coincids with $T_p(N)$, for every $p \in N$. When all integral manifolds of Δ coincide with N, this integral manifold is called a maximal integral manifold.

We refer to Chevally [2] for the proof of the following theorem.

Theorem (1.1.1): Let Δ be an involutive analytic distribution on M. Through every point $p \in M$, there passes a unique maximal integral manifold $T_p(N)$ of Δ .

§ 2. SOME CONCEPTS ON THE THEORY OF LIE GROUPS and LIE ALGEBRAS

Through this section we make use of [2], [4], [5] and [11] .

By a connected complex analytic manifold we mean a complex analytic manifold such that its topological space is connected, or in other words, if it is an arc-wise connected.

Definition (1.2.1): A (connected) complex analytic Lie group, is a group G which is a (connected) complex analytic manifold such that

$$G \times G \longrightarrow G \;;\; (a,b) \longrightarrow ab^{-1}$$

is a complex analytic map.

Now let G be a complex analytic Lie group, then we have :