A STUDY OF A RESERVOIR SUBJECT TO AN INFLOW OF A FINITE NUMBER OF MARKOVIAN COMPONENTS

A Thesis Submitted in Partial
Fulfilment of the Requirents for
The Award of the M. Sc. Degree

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INTRODUCTION

The mathematical models of reservoir behaviour in theory of storage could be of great interest to engineers as they may use their results to reach decisions concerning building and controlling of dams .

The first model is due to Moran (1954) , where he considered a finite dam with inflows forming a stationary (non-seasonal) sequence of mutually independent and identically distributed random variables , release unity , time and volume are discrete , the sequence of reservoir levels then forms a Markov chain , this model is considered as the basis for all subsequent work in this field .

Many results for the corresponding systems with unbounded capacity were followed , such as that given by Kendall (1957) in which he showed that the generating function of the time to first emptiness is given in a form of a functional equation , Prabhu ([illegible]) showed that [illegible] a finite reservoir , the ratio of the probabilities of any two comparable levels is independent of the size of the reservoir , and is in fact the same as the corresponding ratio for the semi-infinite reservoir [illegible] that the generating function of [illegible] distribution of reservoir [illegible] this thesis to models having Markovian inflows.

Let us consider a dam of capacity k , subject to an inflow sequence $\{X_t\}$, $t=o,1,\ldots$, which is assumed to form an ergodic Markov chain , the quantity X_t ($X_t=o,1,\ldots,n$) of water enters during the time interval $(t,t+1)$.

The storage level (dam content) Z_t at time t will be augmented by the inflows X_t , so that there will be an overflow at the time interval $(t,t+1)$ if $X_t+Z_t>k$.

We assume that there will be an instantaneous release R_t from the dam just before time $t+1$, the amount of this release will depend on the content of the dam , this content is equal to $\min(X_t+Z_t,k)$ at the end of the time interval $(t,t+1)$ and just before the release occurs.

We will release M units if $X_t+Z_t\geqslant M$, and release the total content of the dam if it contains less than M units. Hence , the behaviour of the sequence $\{Z_t\}$ of dam contents is controlled by the equation

$$Z_{t+1}=\min(X_t+Z_t,k)-\min(X_t+Z_t,M), \qquad t=o,1,\ldots$$

the corresponding equation for semi-infinite reservoir, $(k\to\infty)$ is

$$Z_{t+1}=X_t+Z_t-\min(X_t+Z_t,M) \qquad t=o,1,\ldots$$

The first chapter deals with the [illegible] [illegible] model , where the inputs form a [illegible] [illegible] capacity is infinite and the release is [illegible] [illegible] We give the generating function of the stationary distribution of levels , proportionality in semi-infinite and finite reservoirs (from which we could get the stationary distribution of storage content of a finite dam) , and the waiting time to first emptiness .

The probability of emptiness in semi-infinite reservoir with general inflow transition matrix , the mean of storage content and some other applications of the generating function of the stationary distribution of levels are also discussed .

Chapter two investigates the generating function of levels , and the waiting time to first emptiness which have been given by Lloyd (9) for the case [illegible] infinite dam subject to a combination of two [illegible] independent Markovian components and unit release. [illegible] present a new result on the probability [illegible] [illegible] the author of this thesis .

In chapter three we generalize Lloyd's [illegible] [illegible] case investigated in chapter two to the case [illegible] reservoir subject to an inflow of a finite [illegible] mutually independent Markovian components , [illegible] a formula for the generating function of levels and discuss the waiting time to first emptiness for this case , all the results presented in this chapter are completely new .

CHAPTER I

SEMI-INFINITE DAM WITH INPUT FORMING A MARKOV CHAIN AND UNIT WITHDRAWALS

INTRODUCTION

This chapter presents the main results and applications of the single stream storage model for semi-infinite reservoir with inputs forming a Markov chain and unit withdrawals , which is more realistic than the classical Moran model where the inputs are assumed to be independent , Lloyd (8) was the first to discuss a model with serially correlated inflows then Odoom and Lloyd (14) extended this work to the case where the inputs are assumed to form a Markov chain , and that was followed by the work done by Khan and Gani (1) , and later by Anis and Lloyd (

We will consider a dam of infinite capacity which is fed during consecutive intervals (t, t+1) by inputs $X_t = 0,1,\dots,n < \infty$, such that each input depends only on the input in the previous time interval.

The storage level Z_t at time t will be augmented by the inflows X_t , now suppose that there is an instantaneous release R_t of water from the dam just before t+1 such that we release a quantity M of water if and a quantity X_t+Z_t if $X_t+Z_t \leq M$, in our discussion will be cocerned with the case of $M=1$.

Thus , the equation that governs the behaviour of the sequence $\{Z_t\}$ of reservoir contents is

$$Z_{t+1} = X_t + Z_t - \min(X_t + Z_t, 1), \qquad t=o,1,\ldots$$

The main results that will be discussed in this chapter are :

(i) The generating function of levels .

(ii) Waiting time to first emptiness for semi-infinite reservoir .

(iii) Proportionality in semi-infinite and finite reservoirs.

1.1- <u>Generating function of levels</u> :

Suppose the input sequence $\{X_t\}$ form an ergodic Markov chain in its limiting equilibrium condition , with

$$\lim_{t\to\infty} P(X_t=r) = p_r , \qquad r=o,1,\ldots,n$$

where

$$E(X_t) = \mu_x < 1 .$$

The $\{p_r\}$ satisfy the linear equations

$$LP = P$$

where $L=(\ell_{rs})$ is the input transition matrix ,

$$\ell_{rs} = p(X_{t+1}=r \mid X_t=s) , \qquad r,s=o,1,\ldots,n$$

It is convenient to partition L into its columns :

$$L = (\ell_o, \ell_1, \ell_2, \ldots, \ell_n) ,$$

π_o	π_1	π_2	π_3	...	π_n	π_{n+1}	π_{n+2}	...	
L_o+L_1	L_o	0	0	...	0	0	0	...	
L_2	L_1	L_o	0	...	0	0	0	...	
L_3	L_2	L_1	L_o	...	0	0	0	...	
...	...	...	...	...	...	...	...	...	(1.1.3)
L_n	L_{n-1}	L_{n-2}	L_{n-3}	...	L_o	0	0	...	
0	L_n	L_{n-1}	L_{n-2}	...	L_1	L_o	0	...	
0	0	L_n	L_{n-1}	...	L_2	L_1	L_o	...	
...	...	...	...	...	...	...	...	...	

It follows from (1,1.3) that

$$\pi_o = (L_o + L_1)\,\pi_o + L_o\,\pi_1$$

$$\pi_r = \sum_{s=o}^{r+1} L_{r+1-s}\,\pi_s \quad , \qquad r=1,2,\ldots$$

If we now introduce the bivariate generating function of the stationary distribution of Z_t and X_t as a vector generating function for the π_r , defining

$$g_{z,x}(\theta) = \sum_r \pi_r \theta^r \quad ,$$

we find that

$$\theta g_{z,x}(\theta) = (\theta - 1)L_o\,\pi_o + H(\theta)g_{z,x}(\theta)$$

$$= (\theta - 1)\ell_o\,\pi_{oo} + H(\theta)g_{z,x}(\theta)$$

whence

$$\{H(\theta) - I\theta\} g_{z,x}(\theta) = \pi_{oo}(1 - \theta)\ell_o \qquad (1.1.4)$$

where I , as usual , denotes the unit matrix .

Premultiply this equation by $1' = (1,1,\ldots,1)$ and utilise the fact that $1'\ell_o = \sum_r \ell_{ro} = 1$, to obtain

$$\pi_{oo}(1 - \theta) = 1'\{H(\theta) - I\theta\} g_{z,x}(\theta) .$$

Now

$$1'H(\theta) = (1,\theta,\theta^2,\ldots,\theta^n) \text{ and } 1'I\theta = (\theta,\theta,\ldots,\theta)$$

whence

$$1'\{H(\theta)-I\theta\} = (1 - \theta)\{1,0,-\theta,-\theta(\theta+1),-\theta(\theta^2+\theta+1),\ldots\}$$

and so

$$\pi_{oo} = \{1,0,-\theta,-\theta(\theta+1),\ldots\} g_{z,x}(\theta)$$

Put $\theta=1$ and note that $g_{z,x}(1) = P$.

Then

$$\pi_{oo} = p_o - \sum_2^n (r-1)p_r$$

$$= 1 - \mu_x$$

whence

$$\pi_{oo} = \lim_{t\to\infty} P(Z_t=o , X_t=o) = 1 - \mu_x \qquad (1.1.5)$$

The marginal distribution of the levels Z_t is given by

$$\lim_{t\to\infty} P(Z_t=r) = \sum_s \pi_{rs} = 1'\pi_r ,$$

and the generating function of this distribution is

$$g_z(\theta) = \sum_r (1'\chi_r)\theta^r = 1' \, g_{z,x}(\theta)$$

$$= (1 - \mu_x)(1 - \theta)1'\{H(\theta) - I\theta\}^{-1} \ell_o , \qquad (1.1.6)$$

by (1.1.4) and (1.1.5). A possibly more convenient form for this may be obtained from the determinantal expression for a bilinear form :

$$X'A^{-1}Y = - \begin{vmatrix} o & X' \\ Y & A \end{vmatrix} / |A| .$$

This gives

$$g_z(\theta) = -(1 - \mu_x)(1 - \theta) \frac{\begin{vmatrix} o & 1' \\ \ell_o & H(\theta)-I\theta \end{vmatrix}}{|H(\theta)-I\theta|} \qquad (1.1.7)$$

EXAMPLE

As an illustration we consider the case of a 3-valued Markov input with matrix

$$L = (\ell_o, \ell_1, \ell_2) = \begin{bmatrix} \ell_{oo} & \ell_{o1} & \ell_{o2} \\ \ell_{1o} & \ell_{11} & \ell_{12} \\ \ell_{2o} & \ell_{21} & \ell_{22} \end{bmatrix} .$$

In this case we find

$$|H(\theta) - I\theta| = \theta^2(1 - \theta)(a\theta - b)$$

where

$$a = \begin{vmatrix} \ell_{11}-1 & \ell_{12} \\ \ell_{21} & \ell_{22} \end{vmatrix} , \qquad b = \begin{vmatrix} \ell_{oo} & \ell_{o1} \\ \ell_{1o} & \ell_{11}-1 \end{vmatrix}$$

and

$$\begin{vmatrix} 0 & 1' \\ \ell_o & H(\theta)-I\theta \end{vmatrix} = -\theta^2(A\theta - B)$$

where

$$= \begin{vmatrix} 1 & 1 & 1 \\ \ell_{1o} & \ell_{11}-1 & \ell_{12} \\ \ell_{2o} & \ell_{21} & \ell_{22} \end{vmatrix} \quad , \quad B = \begin{vmatrix} 1 & 1 \\ \ell_{1o} & \ell_{11}-1 \end{vmatrix}$$

Thus

$$g_z(\theta) = (1 - \mu_x)(A\theta - B)/(a\theta - b)$$

whence

$$p_o = (1 - \mu_x)\,\frac{B}{b}\ ,$$

and the probability that the content is r :

$$p_r = (1 - \mu_x)\left(\frac{B}{b} - \frac{A}{a}\right)\left(\frac{a}{b}\right)^r \ , \quad r=1,2,\ldots$$

<u>The nature of the stationary distribution</u> :

Now (1.1.7) could be written in the form

$$g_z(\theta) = -(1 - \mu_x)(1 - \theta)\ |A(\theta)| / |B(\theta)|$$

where $|B(\theta)|$ is the determinant of order not exceeding n+1, with $B(\theta) = H(\theta) - I\theta = \sum_{0}^{n} L_i\theta^i - I\theta$, and $|A(\theta)|$ is the determinant of order n+2 obtained from this by bordering thus :

$$|A(\theta)| = \begin{vmatrix} 0 & 1' \\ \ell_o & B(\theta) \end{vmatrix}$$

The first point to be noted is that $g_z(\theta)$ is a rational function of θ . We may write $|B(\theta)|$ as $|b_{ij}(\theta)|$, i,j=0,1,...,n

where