STORAGE-MODELS WITH MIXED INPUTS

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by

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It turned out that this case of the two streams is mathematically equivalent (when the inputs are discrete and for a subset of the set of values of the levels of water stored) to the single-stream model with a modified input. Thus the well known results of the single stream model can contribute in solving the problem of a two-stream model without having to start from first principles as Anis and El-Naggar have done. It should be, however, stressed that this baffling and unexpected phenomena, applies only to the case of discrete inputs and even in this case does not apply to the extreme values of the dam's levels, which need separate handling.

Reviewing and commenting on Anis and El-Naggar work, Gani in 1969 [4] gave a shrewed explanation to this phenomena by introducing an artificial random variable V_t which is equal to Y_t when Y_t is less than the release m, and is equal to m when Y_t is greater or equal to m. Although Anis and El-Naggar's work on two-streams model assumes each of the inputs $\{X_t\}$ and $\{Y_t\}$ to be mutually independent and to be independent of each other, the essential point in Gani's argument remains valid whether the inputs are Markovian or independent.

It was thus natural to investigate the case of a single stream model when the input is composed of two independent components, one is serially independent and the other is serially Markovian, because of the relevance of this case to the two-streams model when $\{X_t\}$ is serially Markovian and $\{Y_t\}$ is serially independent and $\{X_t\}$, $\{Y_t\}$ are independent of each other. The investigation for this mixed inputs case was carried by Anis and Lloyd [5].

Later Lloyd [6] extended this work to the case of a single stream model with an input composed of two independent Markovian components. In a sense Anis and Lloyd paper [5] and Lloyd's more recent one [6] are generalisations of Lloyds, 1963, [3] work on the single stream model with a single Markovian component.

Let us now give a brief account of how the two-stream model is seen from the point of view of the single-stream one. As before we assume that the inflows in both streams are discrete. Suppose that the reservoir of finite capacity K is fed by Markovian inflow $\{X_t\}$ and that the flows $\{Y_t\}$ of B, which are independent of those in A, consist of a sequence of mutually independent variables. The object is to maintain the level below the confluence from falling below a fixed value m, by releasing water, when available, to make up any deficit in C. We make also the following assumptions:

a) At time t, the water stored in the dam is Z_t, but $\{X_t\}$ and $\{Y_t\}$ are the flows in A and B respectively during the time interval (t, t+1)

b) Just before the time t+1, an instantaneous release R_t is allowed where R_t being controlled by the input $\{Y_t\}$ of B and such that:

$$R_t = \begin{cases} 0 & \text{if } Y_t \geqslant m \\ \min(Z_t+X_t, i) & \text{if } Y_t = m-i \ , \quad i=1,2,\ldots,m-1 \end{cases}$$

Thus the working equation for Z_{t+1} is given by :

$$Z_{t+1} = \min\ (Z_t + X_t\ , K) - R_t \qquad (1)$$

We now introduce a random variable V_t independent of X_t, and such that:

$$\begin{aligned} V_t &= Y_t && \text{for } Y_t \leqslant m-1 \\ &= m && Y_t \geqslant m \end{aligned}$$

The introduction of this new variable allows as to put R_t in the form :

$$R_t = \min(Z_t + X_t,\ m-V_t)$$

Hence

$$Z_{t+1} = \min(Z_t+X_t, K) - \min(Z_t+X_t,\ m-V_t) \qquad (2)$$

Adding V_t to each element of the right hand side brackets leaves the value of Z_{t+1} unaltered in the equation (2).
Thus:

$$Z_{t+1} = \min(Z_t+X_t' \ , K+V_t) - \min(Z_t+X_t' \ , m) \tag{3}$$

where

$$X_t' = X_t + V_t$$

Now the working equation of the single stream model of a dam with capacity K and release m units is well known as:

$$Z_{t+1} = \min(Z_t+X_t, K) - \min(Z_t+X_t, m) \tag{4}$$

On comparing (3), (4) we note that the two-streams model is equivalent to a single stream one with a modified input $X_t' = X_t + V_t$.

It is true that the capacity in (3) is $K+V_t$ where V_t is random, but it is not difficult to notice that for:

$$0 \leqslant Z_{t+1} \leqslant K-m \qquad (m < K)$$

$$\min(Z_t+X_t', K + V_t) = \min(Z_t + X_t', K)$$

Thus for all: $0 \leqslant Z_{t+1} \leqslant K-m$, (3) becomes :

$$Z_{t+1} = \min(Z_t +X_t', K) - \min(Z_t + X_t' \ , m) \tag{5}$$

which is equivalent to (4).

We thus conclude that for $0 \leqslant Z_{t+1} \leqslant K-m$, the two-streams model with a Markovian input component $\{X_t\}$ and an independent one $\{Y_t\}$ is equivalent to a single stream model with an inflow made up of two components, independent, of each other, one of which is a Markovian sequence and the other is an independent sequence.

The argument may be repeated along the same lines when both inputs $\{X_t\}$ and $\{Y_t\}$ are Markovian.

This brief account shows clearly the importance of dealing first with single-stream model where the inputs are mixed Markovian-independent or where inputs are composed of two Markovian components.

The main part of this thesis deals with these problems in chapters two and three. However, it was thought wise to give in chapter one an account of the well known results (due to Lloyd and Odoom [7], Gani [4] and later to Anis and Lloyd [8]) of the single stream model with a single Markovian inflow component because of the similarities in the approach, and the insight given by the results of chapter one.

Chapter (2) deals with the case of mixed Markovian-independent model. The main theoretical approach is due to Anis and Lloyd [5], but some of the results of this chapter are new, in particular the formulae for the mean storage content in this model, which was not published before.

Chapter (3) expounds the latest work done by Lloyd [5] in the case of two independent Markovian components for the input of the storage model, and in a sense is a generalisation of the previous work done in this.

Chapter (4) deals with a totally different problem namely the wet-period distribution for a finite reservoir with independent inputs. The results of this chapter are totally new, and because of this fact it was thought useful to add them in the last chapter.

ii) The storage of water Z_t at time t will be augmented by the inflow X_t and thus the content will be $\min(K, X_t+Z_t)$ since if $X_t+Z_t > K$ then an amount of water X_t+Z_t-K will overflow.

iii) The withdrawn water, m units (where $m < K$), is supposed to be released in at a uniform rate.

Here we are going to present the main classical results which concern:

i) The distribution of the storage Z_t, when the sequence $\{Z_t\}$ possesses a non-trivial asymptotic equilibrium distribution, for a semi-infinite dam.

ii) The theory of proportionality.

iii) The distribution and the moments of the waiting time to first emptiness for semi-infinite dam.

1.2- Equilibrium distribution of levels :

We shall be concerned, first, with the case of finite reservoir of capacity K, release m units, and a general inflow $\{X_t\}$ forming an ergodic Markov chain assuming the values o,1,2,...,n whose transition probabilities given by:

$$P(X_{t+1}=i|X_t=j) = \ell_{ij} \;; \quad i,j = 0,1,2,\dots,n; \; \sum_i \ell_{ij}=1$$

It is assumed also that this input has a stable limiting distribution given by:

$$\lim_{t \to \infty} P(X_t=i) = p_i \; ; \; \sum_{i=0}^{n} p_i = 1, \; i=0,1,\ldots,n$$

where

$$E(X_t) = \mu_x, \quad \mu_x < 1$$

and the vector $\underline{p}$ is determined by the transition matrix $L = \{\ell_{ij}\}$, such that:

$$L\underline{p} = \underline{p}$$

It has been shown by Lloyd [3] that the behaviour of reservoir level Z_t may be investigated in terms of the joint distribution of the pair (Z_t, X_t) which again forms a Markov-chain so it will be taken from the following treatment :

$$P(Z_{t+1}=r, X_{t+1}=s) = \sum_{r'} \sum_{s'} P(Z_{t+1}=r, X_{t+1}=s \mid Z_t=r', X_t=s') \times P(Z_t=r', X_t=s') \qquad (1.2.1)$$

By the use of the notations:

i) $\pi_t(r,s) = P(Z_t=r, X_t=s)$

ii) $P(r,s \mid r',s') = P(Z_{t+1}=r, X_{t+1}=s \mid Z_t=r', X_t=s')$

equation (1.2.1) becomes:

$$\pi_{t+1}(r,s) = \sum_{r',s'} P(r,s \mid r',s') \, \pi_t(r',s') \qquad (1.2.2)$$

We can arrange the element of $\pi_t(r,s)$ with respect to r and s to form a vector :

$\pi_t = \{\pi_t(0,0)\ \pi_t(0,1), \ldots, \pi_t(0,n);\ \pi_t(1,0),\ \pi_t(1,1), \ldots,$
$\pi_t(1,n);\ \ldots\ ;\ \pi_t(K,0), \ldots, \pi_t(K,n)\}$ which can be partitioned in the form:

$$\pi_t' = (\pi_t'(0),\ \pi_t'(1),\ \ldots,\ \pi_t'(K))$$

where

$$\pi_t(r) = \{\pi_t(r,0),\ \pi_t(r,1), \ldots,\ \pi_t(r,n)\}$$

Thus (1.2.2) can be written in a matrix form as follows:

$$\pi_{t+1} = R\,\pi_t$$

where

$$R = \{P(r,s|r',s')\}$$

But:

$$\begin{aligned} P(r,s|r',s') &= P(Z_{t+1}=r,\ X_{t+1}=s|Z_t=r',\ X_t=s') \\ &= P(Z_{t+1}=r, X_{t+1}=s,\ Z_t=r', X_t=s')/\ P(Z_t=r', X_t=s') \\ &= P(Z_{t+1}=r|Z_t=r', X_t=s')\ P(X_{t+1}=s|Z_t=r', X_t=s') \\ &= \ell_{ss'}\delta_{r\,\omega} \end{aligned}$$

where δ is Kronecker's delta, and :

$$\omega = \begin{cases} 0 & r'+s' \leqslant m \\ r'+s'-m & m < r'+s' < K \\ K & K \leqslant r'+s' \end{cases}$$

and hence:

$$\pi_{t+1}(r,s) = \sum_{s'=o}^{n} \sum_{r'=o}^{K} \ell_{ss'}\, \delta_{r\omega}\, \pi_t(r',s')$$

$$= \sum_{s'=o}^{n} \ell_{ss'}\, \pi_t(r+m-s',s')$$

The Markov transition matrix R governing transitions $\pi_t \longrightarrow \pi_{t+1}$ may be written in the following partitioned form

$\pi_{t+1}(r)$ \ $\pi_t(r)$	r=o	1	...	m-1	m	m+1	...	K-1	K
r=o	$L_{o,m}$	$L_{o,m-1}$		$L_{o,1}$	L_o	0	...	0	0
1	L_{m+1}	L_m		L_2	L_1	L_o		0	0
2	L_{m+2}	L_{m+1}		L_3	L_2	L_1		0	0
	⋮	⋮		⋮	⋮	⋮		⋮	⋮
K-m	L_K	L_{K-1}		L_{K-m+1}	L_{K-m}	L_{K-m-1}		L_1	L_o
⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
K-1	L_{m+K-1}	L_{m+K-2}		L_K	L_{K-1}	L_{K-2}	...	L_m	L_{m-1}
K	$L_{m+K,n}$	L_{m+K-1}		$L_{K,n}$	$L_{K,n}$	$L_{K-1,n}$	...	$L_{m+1,n}$	$L_{m,n}$

(1.2.3)

where L_s is derived from L by retaining the column ℓ_s and replacing all other columns by zeros, so that

$$L = \sum_{s=o}^{n} L_s$$

and where:

$$L_{ij} = L_i + L_{i+1} + \ldots + L_j$$

In the special case of m = 1, the corresponding equation become:

$$\pi_{t+1}(0) = L_{o,1}\,\pi_t(0) + L_o\,\pi_t(1)$$

$$\pi_{t+1}(r) = \sum_{s=o}^{r+1} L_{r+1-s}\,\pi_t(s) \quad , \quad r = 1,2,\ldots,K-1$$

$$\pi_{t+1}(K) = \sum_{s=o}^{K} L_{K+1-s,n}\,\pi_t(s)$$

The asymptotic equilibrium distribution vectors :

$$\pi(r) = \lim_{t\to\infty} \pi_t(r) \quad ; \quad r = o,1,\ldots,K$$

therefore satisfy the relations:

$$\pi(0) = L_{o,1}\,\pi(0) + L_o\,\pi(1)$$

$$\pi(r) = \sum_{s=o}^{r+1} L_{r+1-s}\,\pi(s) \quad ; \quad r = 1,2,\ldots,K-1$$

$$\pi(K) = \sum_{s=o}^{K} L_{K+1-s,n}\,\pi(s)$$

Thus the generalisation of the generating function theorem proceeds as follows: We are concerned with the equilibrium distribution of levels in a semi-infinite reservoir, with unit draft, and with Markovian inflows for which the transition matrix is:

$$L = \sum_{o}^{n} L_r$$