ON SOME PROBLEMS IN COMBINATORICS

THESIS

Submitted in Partial Fulfillment of the Requirements for the Award of M. Sc. Degree

By

SOHEIR MOHAMED SAID MAHMOUD KHAMIS

Submitted at
Ain Shams University
Faculty of Science

1987

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بسم الله الرحمن الرحيم

M.Sc. Courses

Studied by the Author (1983-1984)

at Ain Shams University

1.	Theory of Computation	2 hours weekly for two semesters
2.	Operations Research	2 hours weekly for two semesters
3.	Numerical Analysis	2 hours weekly for two semesters
4.	Programming Language (PASCAL)	2 hours weekly for two semesters
5.	Mathematical Logic	2 hours weekly for one semester
6.	Modern Applied Algebra	2 hours weekly for one semester

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PREFACE

The field of combinatorics has, over the past years, evolved into a fully-fledged branch of discrete mathematics whose potential with respect to computers and natural sciences is only beginning to be realized. Enumeration, (including generating functions, inversion and calculus of finite differences), Configurations, (including designs, permutation groups and coding theory) and Order Theory, (including finite partially ordered sets and lattices), are three large parts from the scope of the field of combinatorics.

In the present thesis we are interested in discussing and solving two problems from order theory. The first one is the problem of counting bipartite H-ordered sets of n points (isomorphism classes of the set of bipartite partially ordered sets), $n \geq 1$. The second one is the problem of counting series-parallel H-ordered sets of n points (isomorphism classes of the set of series-parallel partially H-ordered sets), $n \geq 1$.

The interest in H-ordered sets with specified properties is two-fold. Firstly, H-ordered sets arise naturally in many scheduling problems. In these problems, the tasks to be scheduled are subject to certain partially ordered sets which represent the precedence constraints. Secondly, many order-theoretic and scheduling problems appear in the class of nondeterministic polynomial time complete (NP-complete) problems even in the bipartite case when no task has both predecessors and successors. This explains why a considerable attention was focussed on identifying special classes of precedence constraints for which the problem has an efficient algorithm for solution with a polynomial time. Such classes include the bipartite H-ordered sets and series-parallel H-ordered sets.

The thesis consists of four chapters and an appendix. The necessary background materials are introduced in the first chapter, especially, the interaction between group theory and graph theory. The second chapter is devoted to the counting problem of bipartite H-ordered sets. Our treatment to this problem is based on the use of the results achieved by Riddell [18], Cadogan [6], Harary and Palmer [11] and Hanlon [7] concerning the counting problem for bicolored graphs. Also, some numerical applications and graphical enumerations for some special cases are given. In Chapter III we consider the counting problem of series-parallel H-ordered sets. The derived solution of this problem seems to be original. Numerical applications and graphical enumerations are also considered.

Chapter IV is devoted to the PASCAL programs which are necessary for performing the numerical applications. In the appendix the formulae of the cycle index $Z(S_m \times S_n)$ for the product of the two symmetric groups S_m and S_n with

$p = m + n$ not greater than ten are given.

The article 4 of the second chapter and almost all of Chapter III seem to be original.

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CHAPTER I

SOME BASIC CONCEPTS

1.1 Introduction

This chapter is devoted to the principle definitions and concepts from the theory of groups, theory of graphs and their interactions that are needed in the next chapters.

This chapter consists of an introduction and four sections. In section 1.2 we introduce some definitions from group theory and graph theory. Also, in this section we introduce a number of operations on permutation groups. Section 1.3 is devoted to some results concerning interaction between groups and graphs. In this section we introduce also some results concerning the cycle index formulae for permutation groups. In section 1.4 we discuss Burnside's lemmas since they form the basis for numerous solutions to counting problems for graphs. Section 1.5 is asserted to Polya's enumeration theorem for its basic role in treating the counting problems in graph theory especially for the counting problems where the equivalence classes are determined by two permutation groups.

1.2 Groups and Graphs

This section is devoted to a number of definitions and fundamental operations concerning groups and graphs.

Definition 1.2.1

Let X be a finite set of n elements and let A be a collection of permutations of X which is closed under composition of mappings. Then, A is a permutation group with object set X. The order of A, denoted by $|A|$, is the number of permutations in A and the degree of A is the number n of elements in the object set X, [11].

Definition 1.2.2

A graph G is an ordered pair of disjoint sets (V,E) such that E is a subset of the set of unordered pairs of V. Each element of V is referred to as a vertex and V = V(G) as the vertex set of G and the members of the edge set E = E(G) are called edges, [4].

Definition 1.2.3

An automorphism of a graph G is an isomorphism of G with itself. Each automorphism of G is a permutation of the vertex set V which preserves adjacency. The automorphisms of G form a permutation group, $\Gamma(G)$, which acts on the points of G. It is known as the group of G or as the point-group of G, [10].

Definition 1.2.4

The point-group of a graph G, $\Gamma(G)$, induces another permutation group $\Gamma_1(G)$, called the line-group of G, which acts on the lines of G. Specifically, each permutation

$\alpha\varepsilon\Gamma(G)$ induces a permutation $\alpha^*\varepsilon\Gamma_1(G)$ such that for every element $\{u,v\}\varepsilon E(G)$;

$$\alpha^*\{u,v\} = \{\alpha u,\alpha v\}$$

See [10] and [11].

From the previous definitions, we see that the study of permutation groups evidently goes hand-in-hand with the study of graphs because a graph provides a "picture" of its automorphism group. Thus, the group theoretic concepts required in this thesis are more easily understood in their graph theoretic setting.

Operations on Permutation Groups

There are several important operations on permutation groups which produce other permutation groups. We now state four such binary operations: *product*, *composition*, *exponentiation* and *power group*, (see [10] and [11]).

Let A be a permutation group of order $|A| = m$ and degree d acting on the set $X = \{x_1,x_2,\ldots,x_d\}$, and let B be another permutation group of order $|B| = n$ and degree e acting on the set $Y = \{y_1,y_2,\ldots,y_e\}$.

(1) The product AxB of A and B is a permutation group which acts on the set XxY and whose permutations are all the ordered pairs, written $\alpha x\beta$, of permutations α in A and β in B. The element (x,y) of XxY is permuted by $\alpha x\beta$ as follows:

$$(\alpha x\beta)(x,y) = (\alpha x,\beta y). \qquad (1.2.1)$$

(2) The composition A[B] of "A around B" also acts on XxY. For α in A and sequence $(\beta_1, \beta_2, \ldots, \beta_d)$ of d (not necessarily distinct) permutations in B, there is a unique permutation in A[B] written $[\alpha; \beta_1, \beta_2, \ldots, \beta_d]$ such that for (x_i, y_j) in XxY; $1 \leq i \leq d$, $1 \leq j \leq e$:

$$[\alpha; \beta_1, \beta_2, \ldots, \beta_d](x_i, y_j) = (\alpha x_i, \beta_i y_j). \qquad (1.2.2)$$

(3) The power group denoted by B^A has the collection Y^X of functions from X into Y as its object set. The permutations of B^A consist of all ordered pairs, written $(\alpha;\beta)$, of permutations α in A and β in B. The image of any $f \varepsilon Y^X$ under $(\alpha;\beta)$ is given by

$$((\alpha;\beta)f)(x) = \beta f(\alpha x), \qquad (1.2.3)$$

for each x in X.

(4) The exponentiation $[B]^A$ of A with B is that a permutation group with object set Y^X whose permutations are constructed as follows. Each permutation α in A and each sequence $\beta_1, \ldots, \beta_d$ of permutations in B determine just one permutation $(\alpha; \beta_1, \ldots, \beta_d)$ in $[B]^A$, which takes the element $f \varepsilon Y^X$ into the element $f^* \varepsilon Y^X$ defined for all $x_i \varepsilon X$ by

$$f^*(x_i) = \beta_i f(\alpha x_i) \quad . \qquad (1.2.4)$$

These operations are necessary for finding a permutation group of a composite graph that formed from other graphs by various operations. In Chapter II we will use these operations to obtain the generating function for bicolored graphs.