ON THE DEFINITE INTEGRAL SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS OF ORDER k.

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By

ABD EL-SATTAR ABDEL-HAMID IBRAHIM DABOUR

Department of Pure Mathematics
Faculty Of Science
Ain Shams University
Abbasia, Cairo

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A. A. DABOUR

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C O N T E N T S

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CHAPTER I

ON THE DEFINITE INTEGRAL SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(k)} - x y = 0$

In this Chapter, we discuss briefly two different methods to obtain the definite integral solutions of the differential equation

$$y^{(k)} - x y = 0 .$$

The first method is the usual one [1] which was used by Z. Mursi and K. R. Yacoub [2] to obtain the definite integral solutions of the differential equation

$$y^{(k)} - x y = 0 .$$

The second method is that used later on by K.R. Yacoub and A. R. Yacoub [3] to obtain the definite integral solutions of the differential equation

$$y^{(k)} + x y = 0 .$$

The fundamental idea which underlies the first method is the assumption of the definite integral solution in the form

$$y = \int e^{xt} \psi(t)\, dt,$$

then the insertion of this integral in the differential equation under consideration in order to determine the form of the function $\psi(t)$ and the limits of integration.

In the second method, the assumed integral has been changed to the form

$$y = \int e^{wxt} \psi(t)\, dt,$$

where w is in general a complex quantity.
The merit of this modification is that it gives in a systematic manner two complete classes of definite integral solutions.

In the first method, just one class of definite integral solutions has been furnished. However, in the second method, a proper choice of the complex number w simplifies the definite integral solutions obtained to those of the first method.

1. DESCRIPTION OF THE PROPOSED METHOD

To solve the differential equation

$$y^{(k)} - x\,y = 0\,, \qquad (1)$$

assume that,

$$y = \int e^{wxt}\,\psi(t)\,dt,$$

where w is in general a complex quantity and $\psi(t)$ is a function of the real variable t but not of x, the form of this function and the limits of integration (supposed independent of x) are to be determined by inserting the assumed integral in the differential equation (1). We thus have

$$w^{k}\int t^{k}\,e^{wxt}\,\psi(t)\,dt - x\int e^{wxt}\,\psi(t)\,dt = 0.$$

Integrating by parts,

$$\int x\,e^{wxt}\,\psi(t)\,dt = \frac{1}{w}\int \psi(t)\,\frac{d}{dt}\,e^{wxt}\,dt$$

$$= \left[\frac{1}{w}\,\psi(t)\,e^{wxt}\right] - \frac{1}{w}\int e^{wxt}\,\frac{d}{dt}\,\psi(t)\,dt.$$

Thus we have

$$w^k \int t^k \; e^{wxt} \psi(t)\, dt - \left[\frac{1}{w} \psi(t)\, e^{wxt} \right] +$$

$$\frac{1}{w} \int e^{wxt} \; \frac{d}{dt} \psi(t)\; dt \; = \; 0,$$

where the middle term is taken between the limits of integration as yet unknown.

The last equation can be written

$$\int e^{wxt} \left\{ w^k \; t^k \psi(t) + \frac{1}{w} \frac{d}{dt} \psi(t) \right\} dt$$
$$- \left[\frac{1}{w} \; e^{wxt} \psi(t) \right] \; = \; 0.$$

This equation will be satisfied if

$$\frac{1}{w} \; \frac{d}{dt} \psi(t) + w^k \; t^k \psi(t) \; = \; 0 \; , \qquad (2)$$

for all values of t within the range of integration, and if

$$\left[\frac{1}{w} \; e^{wxt} \psi(t) \right] \; = \; 0 \qquad (3)$$

between the limits. Equation (2) determines $\psi(t)$ as a function of t while equation (3) will determine the limits of the assumed integral. Equation (2) can be written in the form

$$\frac{d}{dt}\psi(t) = -w^{k+1} t^{k} \psi(t),$$

which gives directly

$$\psi(t) = C \exp\left\{-w^{k+1} t^{k+1}/(k+1)\right\},$$

where C is an arbitrary constant. Then the equation determining the limits will take the form

$$\left[\frac{C}{w} \exp\left\{w x t - w^{k+1} t^{k+1}/(k+1)\right\}\right] = 0.$$

We may take 0 and ∞ as the limits of the definite integral so that the left hand side of the last equation reduces to $-\frac{C}{w}$ provided that

$$\left.\begin{array}{ll} & \text{(i) } RI\ w^{k+1} > 0, \\ \text{or} & \\ & \text{(ii) } RI\ w^{k+1} = 0 \text{ and } x\ RI\ w < 0. \end{array}\right\} \quad (4)$$

Writing

$$F_k(x,w) = \int_0^\infty \exp\left\{wxt - w^{k+1} t^{k+1}/(k+1)\right\} dt, \quad (5)$$

we readily see that the definite integral $C\,w\,F_k(x,w)$ is a particular integral of the equation

$$y^{(k)} - x\,y = C,$$

where w satisfies any of the conditions (4.i) or (4.ii).

It will be seen that associated with (4.i) or (4.ii), there corresponds a complete class of definite integral solutions.

2. THE FIRST CLASS OF DEFINITE INTEGRAL SOLUTIONS

This class is characterized by the choice of a complex number w for which $RI\ w^{k+1} > 0$. For the sake of simplicity of the form of the solution, we take

$$w^{k+1} = 1$$

However, the form of the primitive will differ according as k is even or odd, for this reason we consider each case seperately.

CASE I. $k = 2n$.

In this case, the differential equation is

$$y^{(2n)} - x\,y = 0\,, \qquad (6)$$

and the corresponding values of w are given by $w^{2n+1} = 1$, so that

$$w = 1\,, \quad e^{\pm i\theta r}\,,$$

where

$$\theta_r = \frac{2r}{2n+1}\pi \quad , \quad r = 1, 2, \ldots\ldots, n$$

It may be remarked that $e^{i\theta_0} = e^{-i\theta_0} = 1$. Then with the notation introduced in (5), we see that

$$y = \sum_{r=0}^{n} C_r \, e^{i\theta_r} \, F_{2n}(x, e^{i\theta_r}) \qquad (7)$$

where C_0, C_1, ..., C_n are arbitrary constants, is a solution provided that

$$\sum_{r=0}^{n} C_r = 0 \qquad (8)$$

Remembering that $e^{i\theta_0} = 1$ and substituting for C_0 its value given by (8), then (7) gives

$$y = \sum_{r=1}^{n} C_r \, e^{i\theta_r} \, F_{2n}(x, e^{i\theta_r}) + C_0 F_{2n}(x,1)$$

$$= \sum_{r=1}^{n} C_r \, e^{i\theta_r} F_{2n}(x, e^{i\theta_r}) - \sum_{r=1}^{n} C_r F_{2n}(x,1)$$

i.e. $$y = \sum_{r=1}^{n} C_r \, e^{i\theta_r} \left\{ F_{2n}(x, e^{i\theta_r}) - e^{-i\theta_r} F_{2n}(x,1) \right\} .$$

It is readily seen that

$$G_{2n}(x, e^{i\theta_r}) = F_{2n}(x, e^{i\theta_r}) - e^{-i\theta_r} F_{2n}(x,1):$$

$r = 1, 2, 3, \ldots, n,$

form n distinct solutions of the differential equation (6). In a similar way

$$G_{2n}(x, e^{-i\theta_r}) : \quad r = 1, 2, \ldots, n$$

form other n distinct solutions. With the notation introduced in (5), we see that

$$y = \sum_{r=o}^{n} C_r \, e^{-i\theta_r} F_{2n}(x, e^{-i\theta_r}) \tag{9}$$

where C_o, C_1, ..., C_n are arbitrary constants is a solution provided that

$$\sum_{r=o}^{n} C_r = 0 \quad . \tag{10}$$

From (10) we have

$$C_o = - \sum_{r=1}^{n} C_r$$

Then (9) gives

$$y = \sum_{r=1}^{n} C_r e^{-i\theta_r} F_{2n}(x, e^{-i\theta_r}) + C_0 F_{2n}(x,1)$$

$$= \sum_{r=1}^{n} C_r e^{-i\theta_r} F_{2n}(x, e^{-i\theta_r}) - \sum_{r=1}^{n} C_r F_{2n}(x,1)$$

$$= \sum_{r=1}^{n} C_r e^{-i\theta_r} \left\{ F_{2n}(x, e^{-i\theta_r}) - e^{i\theta_r} F_{2n}(x,1) \right\}.$$

It is readily seen that

$$G_{2n}(x, e^{-i\theta_r}) = F_{2n}(x, e^{-i\theta_r}) - e^{i\theta_r} F_{2n}(x,1):$$

$r = 1, 2, \ldots, n$, form n distinct solution of the differential equation (6).

In conclusion the solutions

$$G_{2n}(x, e^{\pm i\theta_r}) : \quad r = 1, 2, \ldots, n,$$

furnish 2n distinct solutions for the differential equation (6).

Moreover if we write

$$G_{2n}(x, e^{i\theta_r}) = Ci_{2n}(x, \theta_r) + i\, Si_{2n}(x, \theta_r)$$

We see that $Ci_{2n}(x,\theta_r)$ and $Si_{2n}(x,\theta_r)$ are distinct solutions of (6).

Furthermore

$$G_{2n}(x,\ e^{i\theta_r}) = F_{2n}(x,\ e^{i\theta_r}) - e^{-i\theta_r} F_{2n}(x,\ 1)\ ,$$

$$= \int_0^{\infty} \exp\left\{xt\ e^{i\theta_r} - e^{i(2n+1)\theta_r}\ t^{2n+1} / (2n+1)\right\} dt$$

$$- e^{-i\theta_r} \int_0^{\infty} \exp\left\{x\ t - t^{2n+1} / (2n+1)\right\} dt,$$

i.e.

$$Ci_{2n}(x,\ \theta_r) + i\ Si_{2n}(x,\ \theta_r) =$$

$$= \int_0^{\infty} \exp\left\{x\ t\ (\cos\theta_r + i\sin\theta_r) - \frac{t^{2n+1}}{2n+1}\right\} dt$$

$$- (\cos\theta_r - i\sin\theta_r) \int_0^{\infty} \exp\left\{x\ t - \frac{t^{2n+1}}{2n+1}\right\} dt,$$

$$= \int_0^{\infty} \exp\left\{x\ t\ \cos\theta_r - \frac{t^{2n+1}}{2n+1}\right\} .$$

$$. \left\{\cos(xt\sin\theta_r) + i\sin(xt\sin\theta_r)\right\} dt$$

$$- (\cos \theta_r - i \sin \theta_r) \int_0^{\infty} \exp \left\{ xt - \frac{t^{2n+1}}{2n+1} \right\} dt.$$

Thus we have the following.

<u>Conclusion 1:</u>

The definite integrals

$$Ci_{2n}(x, \theta_r) =$$

$$= \int_0^{\infty} \exp \left\{ xt \cos \theta_r - \frac{t^{2n+1}}{2n+1} \right\} \cos (xt \sin \theta_r) \, dt$$

$$- \cos \theta_r \int_0^{\infty} \exp \left\{ x t - \frac{t^{2n+1}}{2n+1} \right\} dt, \qquad (11)$$

and

$$Si_{2n}(x, \theta_r) =$$

$$= \int_0^{\infty} \exp \left\{ x t \cos \theta_r - \frac{t^{2n+1}}{2n+1} \right\} \sin (xt \sin \theta_r) \, dt$$

$$+ \sin \theta_r \int_0^{\infty} \exp \left\{ x t - \frac{t^{2n+1}}{2n+1} \right\} dt, \qquad (12)$$

where $\theta_r = \frac{2r}{2n+1}\pi$ and $r = 1, 2, \ldots, n$ furnish $2n$ real distinct solutions for the differential equation

$$y^{(2n)} - x y = 0 .$$