SOME MIXED BOUNDARY-VALUE PROBLEMS OF THE DIRICHLET - NEWTON TYPE

A Thesis
submitted in partial fulfillment
of the requirements
of the award of the
MASTER OF SCIENCE degree

by

HEMEIDA EL-SAYED A. GAD-ALLAH

Supervisors:

PROF. DR. AHMED G. EL-SAKKA
Professor of Mathematics
Faculty of Science
Ain Shams University

DR. M. GAMAL EL-SHEIKH
Lecturer of Mathematics
Faculty of Science
Ain Shams University

submitted to
Ain Shams University
Faculty of Science
Department of Mathematics

CAIRO 1986

بسم الله الرحمن الرحيم

بسم الله الرحمن الرحيم

TO PARENTS

NOTE

The present thesis is submitted to Ain Shams University in partial fulfillment of the requirements of the degree of Master of Science in Applied Mathematics.

Besides the research work materialized in this thesis, the candidate has attended six postgraduate courses within a year (1982 - 1983) including the following topics:

(1) Theory of Elasticity
(2) Theory of Stability
(3) Fluid Mechanics
(4) Classical Mechanics
(5) Linear System Analysis
(6) Numerical Analysis.

The applicant HEMEIDA ELSAYED AHMED GAD-ALLAH has successfully passed the final examination of these courses.

ACKNOWLEDGEMENTS

I am very indebted to my supervisor Prof.Dr. A. G. EL-SAKKA, Professor of Mathematics, Faculty of Science, Ain Shams University, for his good advice .

I would like to express my deep appreciation to my supervisor Dr. M. GAMAL EL SHEIKH, Lecturer of Mathematics, Faculty of Science, Ain Shams University, for suggesting the problem and for his constructive guidance and warm encouragement throughout his supervision of this work.

Many thanks are also due to Prof. Dr. M. A. KHIDR, head of the Department of Mathematics, Faculty of Science, Ain Shams University, for his encouragement and hospitality.

I am very grateful to Prof. Dr. EL-SAYED M. EL-GHAZZY for his encouragement throughout this work.

Contents

Summary 1

Chapter I

Introduction

1.1 The Cauchy type integrals with Hölder-continuous density 3
1.2 The Sokhotski formulae 5
1.3 The finite Fourier transform 7
1.4 Chebyshev polynomials 11
1.5 The wing equation 13
1.6 Infinite systems of algebraic equations 14

Chapter II

Illustration of the method by means of a simple problem

2.1 Dirichlet-Newton problem for the Laplace equation in the circle and its reduction to a discrete problem 20
2.2 The transformation of the discrete problem to the wing equation 21
2.3 The transformation of the wing equation to the algebraic system 24
2.4 Regularity condition of the algebraic systems 29
2.5 Conversion to the original variable θ 34
2.6 Other problems 36

Chapter III

Dirichlet - Newton problems with two mixed boundary conditions

3.1 A mixed boundary value problem for the stationary heat equation in the annulus 40
3.2 Calculation of the integrals $I_{n,2k}$ 47

CHAPTER IV

NUMERICAL EXAMPLES

4.1 Numerical calculations for the problem with one mixed condition 53

4.2 Numerical calculations for the problem with two mixed conditions 58

REFERENCES 69

ARABIC SUMMARY

SUMMARY

SUMMARY

In 1961, Ju. I. Cherski [1] started to develop a method for solving steady mixed boundary-value problems of the "Dirichlet-Neumann" type. Gradually, this method has become of wide application for solution of problems in several branches of mathematical physics [2], [3]. In brief, this method reduces the mixed boundary - value problems to systems of algebraic equations.

In 1981, M. G. El-Sheikh [4] modified the above method to solve initial value problems with mixed boundary conditions. The procedures of this modification suggest the validity of the statement that the methods in which a problem is reduced to a system of algebraic equations have the advantage that they can be extended to include initial value problems with mixed boundary conditions identical to those in the stationary problem.

This work presents a special method for solving steady problems with " Dirichlet - Newton " boundary conditions. The proposed method consists of reducing the problem to a discrete problem by means of the finite Fourier transform in a way like that followed in [1]—[4]. This in turn is transformed to the aircraft wings singular integro-differential equation. Using the orthogonal Chebyshev polynomials [5], the latter equation can be reduced to an infinite system of algebraic equations.

The thesis consists of four chapters. The first chapter is an introductory one. It exhibits all the mathematical background

and tools on which the method is based.

In the second chapter, the method is illustrated by means of a typical problem: the stationary heat equation within the unit circle. It has been shown that the algebric system, to which the problem is reduced, is always quasi-regular. In other words , this system can, in principle, be solved for all values of the parameters in the problem . Moreover, the domain of regularity of the system is defined. Finally, more examples of simple problems which can be solved by the method are outlined.

It has been shown in chapter three that the method can be extended to reduce problems with several mixed conditions to several algebraic systems. In order to complete the solution of such problems, a class of integrals are specially calculated.

In the last chapter, the procedures are carried out right to the numerical step. The numerical results suggest stronglv that the method of truncation is convenient for solving algebraic systems to which simple problems are reduced.

The more the problem becomes of more complicated geometry, the higher the order of the truncation should be considered, and one will have to refer to a high speed computer.

CHAPTER I

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1. THE CAUCHY TYPE INTEGRALS WITH HOLDER-CONTINUOUS DENSITY.

The integral

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\phi(\tau)}{\tau - z} \, d\tau \qquad (1.1.1)$$

, where L is a smooth contour in the z-plane and $\phi(\tau)$ is a continuous function on L , is called the Cauchy type integral. By a smooth contour we understand a simple , closed or open line with continuously varying tangent and having no-recurrent points.

The Cauchy type integral (1.1.1) defines a function which is analytic in the entire plane except for the points of L . Further, it vanishes at infinity; i.e.

$$\Phi(\infty) = 0 . \qquad (1.1.2)$$

This result can easily be verified by expanding the kernel of the Cauchy type integral in the vicinity of infinity into the series

$$\frac{1}{\tau - z} = -\frac{1}{z} - \frac{\tau}{z^2} - \dots - \frac{\tau^{n-1}}{z^n} - \dots,$$

in which the zero power is absent. Also , it is well-known that if the density $\phi(z)$ is analytic inside L and continuous on L, then

$$\Phi(z) = \begin{cases} \phi(z) , & z \in D^+ , \\ 0 , & z \in D^- , \end{cases} \qquad (1.1.3)$$

D^+ is the Domain within L and D^- is that outside L . However, if $\phi(z)$ is analytic outside L and continuous on L , then

$$\Phi(z) = \begin{cases} \phi(\infty) & , z \in D^+ \\ -\phi(z) + \phi(\infty) & , z \in D^- . \end{cases} \qquad (1.1.4)$$

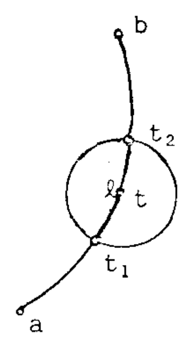
As the point z approaches the contour L, the Cauchy type integral can still be defined for the densities satisfying the Hölder-condition on L . By definition $\phi(t)$ obeys the Hölder-condition if the relation

$$|\phi(t_2) - \phi(t_1)| < A |t_2 - t_1|^{\lambda}, \quad 0<\lambda<1 \qquad (1.1.5)$$

is satisfied for any two points t_1 and t_2 on L . In this case the Cauchy principal value of the curvilinear integral

$$\Phi(t) = \int_L \frac{\phi(\tau)}{\tau - t} d\tau \qquad (1.1.6)$$

, where τ and t are complex coordinates of points on the smooth contour L, can be defined. To this end, we describe a circle of radius ρ and centre t on L and denote t_1 and t_2 the points of the circle intersecting L. The Cauchy principal value of the singular integral (1.1.6) is defined as the limit of the integral



$$\int_{L-\ell} \frac{\phi(\tau)}{\tau - t} d\tau = \int_{L-\ell} \frac{\phi(\tau) - \phi(t)}{\tau - t} d\tau + \phi(t) \int_{L-\ell} \frac{d\tau}{\tau - t}$$

as ρ tends to zero , ℓ is the part t_1t_2 of L. In view of the Hölder condition (1.1.5), the first integral exsists as improper and the second is equal to $\ell n\{(b-t)/(a-t)\} + i\pi$. Thus , integral (1.1.6) can be represented in the form

$$\Phi(t) = \int_L \frac{\phi(\tau)}{\tau - t} d\tau = \int_L \frac{\phi(\tau) - \phi(t)}{\tau - t} d\tau + \phi(t)\, \ell n \frac{b-t}{t-a} \qquad (1.1.7)$$

in which the branch of the logarithmic function is selected in such a way that $\ell n(-1) = \pi i$. In particular, if L is closed, equation (1.1.7) becomes

$$\int_L \frac{\phi(\tau)}{\tau - t} d\tau = \int_L \frac{\phi(\tau) - \phi(t)}{\tau - t} d\tau + i\pi\, \phi(t) . \qquad (1.1.8)$$

1.2. THE SOKHOTSKI FORMULAE.

If the density $\phi(\tau)$ obeys the Hölder condition,two relations can be derived connecting the singular integral (1.1.6) and the limiting values $\Phi^{\pm}(t)$ of the Cauchy type integral (1.1.1):

$$\Phi^{\pm}(t) = \lim_{z \to t^{\pm}} \frac{1}{2\pi i} \int_L \frac{\phi(\tau)}{\tau - z} d\tau . \qquad (1.2.1)$$

The symbol $t^+(t^-)$ means that z approaches t on a path lying to the left (right) of L. This connection follows from the continuity of the function

$$\Psi(z) = \int_L \frac{\phi(\tau) - \phi(z)}{\tau - z} d\tau \qquad (1.2.2)$$

at all points of the contour L except its end points [6] . In the notation of (1.2.1) this yields :