

STABILITY IN DYNAMICAL
SYSTEMS

THESIS

SUBMITTED IN PARTIAL FULFILLMENT

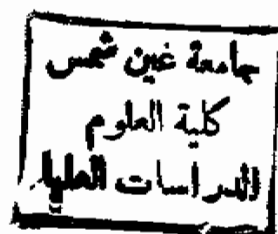
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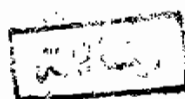
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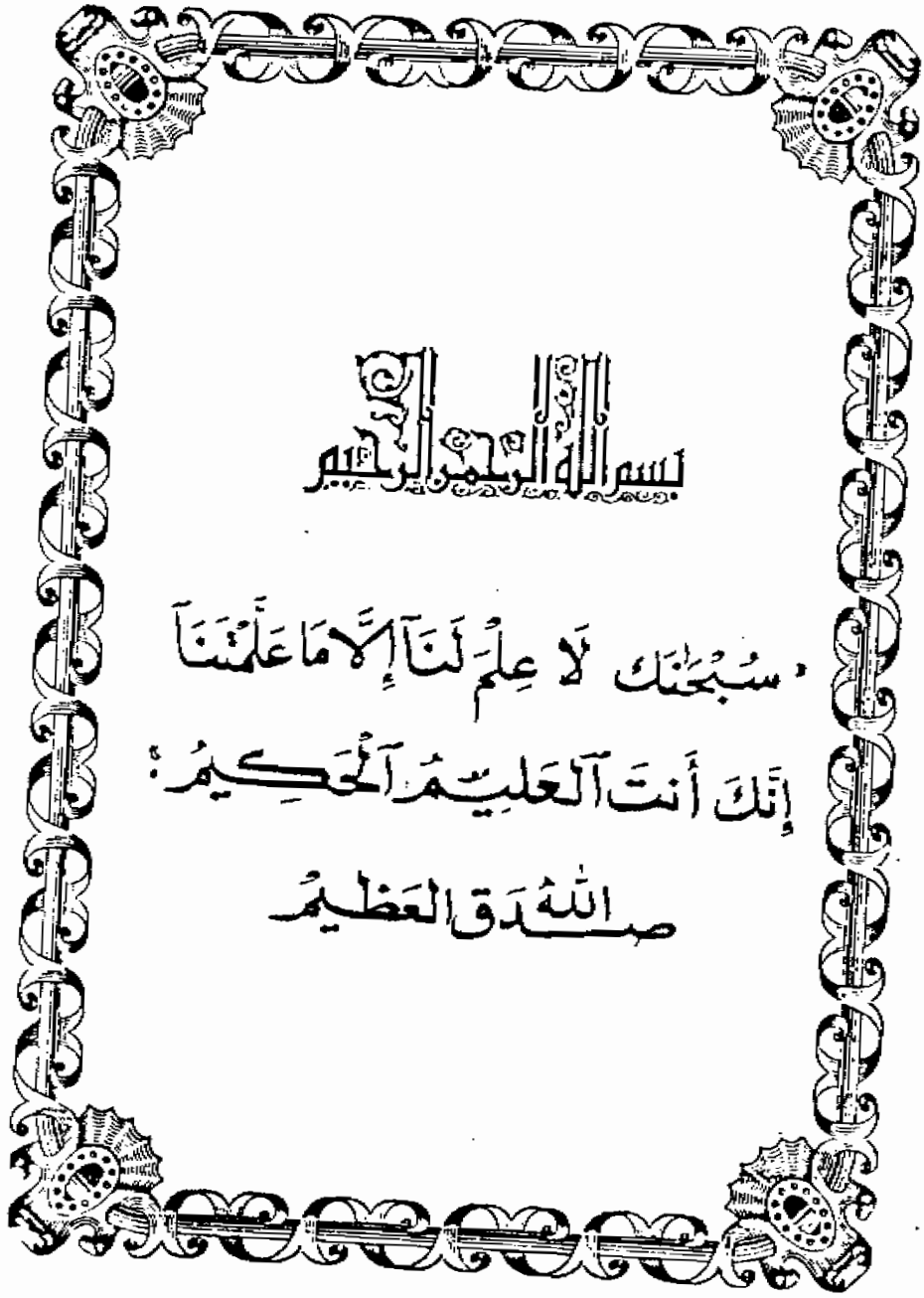
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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَا
إِنَّكَ أَنْتَ الْعَلِيمُ الْحَكِيمُ
صَلِّ عَلَى مُحَمَّدٍ الْعَظِيمِ



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PREFACE

Dynamical systems generally speaking is the study of transformations which preserves a certain measure. Abstractly one has a space X and a transformation π of X (or a family of transformations π_t , $t \in T$ where T is a given group) such that π (or π_t) preserves the measure defined on X . The nature of any work on dynamical systems can be classified depending on the nature of both X and T . For example, if X is just a topological space we have a topological dynamics. If T is the real line $\mathbb{R}$ we have a continuous dynamical system. If T is $\mathbb{Z}$ then we have a discrete dynamical system. In the case where X is a differential manifold with $T = \mathbb{R}$ we have a continuous flow and so on.

The aim of this work is to study the stability properties of dynamical systems where X is a topological or a metric space.

This work consists of four chapters. In the first chapter we study dynamical systems on topological spaces and we discuss some properties of these systems. Also, in this chapter we discuss minimal sets on compact dynamical systems and the limit sets Ω_X and A_X of trajectories. In the second chapter we discuss the different types of stability ("Lagrange and Liapunov Stability) in a dynamical system.

Also we prove one of the most important properties of Poisson stability. In the third chapter we discuss the mean properties of attractors and we also give some examples to illustrate different types of attractors. In the fourth chapter we shall define what is meant by a perturbed orbit, then we shall define a mapping on a set of dynamical systems using this notation and finally study the properties of this mapping. We also define a class of dynamical systems and prove that their perturbed orbits are close to periodic orbits.

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CHAPTER I

DEFINITIONS & ELEMENTARY PROPERTIES

Introduction

In this chapter we discuss the main definitions of dynamical systems. The chapter consists of four articles. In article one we state the definitions of topological and metric spaces which we need throughout the thesis. In the second article we give the basic definitions of dynamical system on topological spaces. We discuss some properties of these systems and also some properties of invariant sets in dynamical system. In the third article we discuss minimal sets on compact dynamical system. Finally in article four we discuss the limit sets Ω_x and A_x of trajectories. The main references for this chapter are: [7,8,9,15]

1.1 Basic Definitions (Topological and Metric Space)

Definition 1.1.1

A topological space is a set X and a class of subsets of X called open sets of X , such that the class contains $\emptyset$ and X and is closed under the formation of finite intersections and arbitrary unions.

Definition 1.1.2

The interior, E^O , of a subset E of X is the union of all open sets contained in E , the closure, $\bar{E}$, of E is the intersection of all closed set containing E . If E is open

then $E^{\circ} = E$, and, if E is closed, then $\bar{E} = E$.

Definition 1.1.3

A set E is dense in X if $\bar{E} = X$.

Definition 1.1.4

A space X is separable if it has a countable dense set E .

Definition 1.1.5

A collection of open sets $\{G_{\alpha}\}$ is called an open cover of a set K if $\bigcup_{\alpha} G_{\alpha} = K$.

Definition 1.1.6

A set E in X is compact if for every open cover K of E there exists a finite subclass $\{K_1, K_2, \dots, K_n\}$ of K which is an open covering of E .

Definition 1.1.7

A topological space is a Hausdorff space if every pair of distinct points has disjoint neighborhoods.

Definition 1.1.8

A transformation T from a topological space X into a topological space Y is continuous if the inverse image of every open set is open, or, equivalently, if the inverse image of every closed set is closed.

Definition 1.1.9

If $T: X \rightarrow X$ is a continuous invertible transformation such that T^{-1} is also continuous then we say that T is a homeomorphism.

Definition 1.1.10

The transformation T is open if the image of every open set is open.

Definition 1.1.11

A metric space is a set X and real valued function d (called distance) is on $X \times X$, such that

- (i) $d(x,y) \geq 0$
- (ii) $d(x,y) = 0$ iff $x = y$, $d(x,y) = d(y,x)$
- (iii) $d(x,y) \leq d(x,z) + d(z,y)$.

1.2 Basic Definitions and Elementary Properties of Dynamical Systems

Definition 1.2.1

Let X be a topological space. Let T be an additive topological group. Then, a transformation $\pi: X \times T \rightarrow X$ is said to define a dynamical system (X, T, π) on X if it has the following properties:

- (i) $\pi(x, 0) = x$ for all $x \in X$
- (ii) $\pi(\pi(x, t), s) = \pi(x, t+s)$ for all $x \in X$ and all $t, s \in T$
- (iii) π is continuous.

For every $t \in T$ the mapping π induces a continuous mapping $\pi^t: X \rightarrow X$ such that $\pi^t(x) = \pi(x, t)$.

The mapping π^t is called transition.

Theorem 1.2.2

The mapping π^{-t} defined by $\pi^{-t}(x) = \pi(x, -t)$ is the inverse of the mapping π^t .

Proof

It must be proved $(\pi^t)^{-1} = \pi^{-t}$. This can be easily shown by applying to the point $x \in X$ the mapping π^t , then to the point $y = \pi(x, t)$ we apply π^{-t} . Let $z = \pi^{-t}(y)$, then z must coincide with x . In fact, using axioms (i) and (ii) we have

$$z = \pi^{-t}(\pi(x, t)) = \pi(\pi(x, t), -t) = \pi(x, t-t) = \pi(x, 0) = x$$

which proves the theorem.

Theorem 1.2.3

The mapping π^t is a homeomorphism where π^{-t} is the inverse of π^t .

Proof

The mapping π^t is an onto mapping. In fact, all points $x \in X$ are image points of points $\pi(x, -t) \in X$. For the same reasons the mapping π^t is one to one. In fact the statement $\pi(x, t) = \pi(y, t) = z$, where $x, y, z \in X$, $t \in T$ are fixed implies, by applying π^{-t} , that $x = y = \pi(z, -t)$, which shows that π^t is one to one.

Since, by 1.2.1, π^{-t} is obviously continuous, the theorem is proved.

Note 1.2.4

We note that the set $\{\pi^t\}$, $t \in T$ is a group with the group operation defined by

$$\pi^t \pi^s = \pi^{t+s}$$

where π^0 is the identity element and for any π^t , π^{-t} is the inverse. In fact,

$$(i) \quad \pi^t \pi^0 = \pi^{t+0} = \pi^t$$

$$(ii) \quad \pi^t \pi^{-t} = \pi^{t-t} = \pi^0$$

$$(iii) \quad \pi^t (\pi^s \pi^q) = \pi^t \pi^{s+q} = \pi^{t+(s+q)} = \pi^{(t+s)+q} = \pi^{t+s} \pi^q \\ = (\pi^t \pi^s) \pi^q.$$

Notice also that the group is commutative.

$$(iv) \quad \pi^t \pi^s = \pi^{t+s} = \pi^{s+t}.$$

A Simplified Notation

We shall write x_t or $x(t)$ instead of $\pi(x, t)$ for a fixed t , x_t is, therefore, the image of the point $x \in X$ under the mapping π^t . In this simplified notation we see that

$$x_0 = x \text{ and } (x_t)_s = x_{t+s}$$

Definition 1.2.5

For every $x \in X$ the trajectory orbit through x is the set

$$xT = \{x(t), t \in T\}$$

Definition 1.2.6

In the case where $T = \mathbb{R}$ the sets xR^+ and xR^- defined by $xR^+ = \{x(t), t \in \mathbb{R}^+\}$ and $xR^- = \{x(t), t \in \mathbb{R}^-\}$ are called respectively the positive and negative semi-trajectory (orbit) through x .

The set $x[a, b]$ defined by $x[a, b] = \{x_t, t \in [a, b]\}$ is called a trajectory segment.

If $T = \mathbb{R}$ we say that we have a continuous dynamical system. If $T = \mathbb{Z}$ we say that we have a discrete dynamical system.

Note 1.2.7

(1) It is clear that $xR = (x_t)R, t \in \mathbb{R}$.

(2) From the properties of π_x it follows that the trajectory segment is a closed bounded set.

Definition 1.2.8

A point $x \in X$ such that $xT = \{x\}$ is called a critical or fixed point. It is clear that this point is a fixed point for the mapping $\pi^t: X \rightarrow X$. It is also called stationary or equilibrium or rest point.

Theorem 1.2.9

Let (X, T, π) be a continuous or discrete dynamical system. If for $a < b, a, b \in T, x \in X$ we have

$$x[a, b] = \{x\} \quad (1)$$

then x is a fixed point.