

**NUMERICAL SOLUTIONS OF INITIAL-
BOUNDARY-VALUE PROBLEMS
FOR PARTIAL DIFFERENTIAL
EQUATIONS**

1.95.15

THESIS

**SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS
FOR THE AWARD
OF THE (M. Sc.) DEGREE**



30494

BY

ISMAIL KAOUD ABD EL-AZIZ

515.35
I.H

SUPERVISED BY

Prof. Dr. E. El Shobaky

Prof. Dr. H. Nasr

**SUBMITTED TO
FACULTY OF SCIENCE
AIN SHAMS UNIVERSITY
(CAIRO)**

Handwritten signature or initials.

1989

9

ACKNOWLEDGEMENT

I am deeply grateful to Prof. Dr. E. El Shobaky Professor of Pure Mathematics, Faculty of Science, Ain Shams University and to Major General Prof. Dr. Hassan Nasr for suggesting the point of research, for continuous guidance, encouragement, helpful comments, invaluable suggestions and capable supervision.

I am also thankfull to Prof. Dr. M. A. Khidr, Head of the Department of Mathematics for his encouragement and help.



M. Sc. COURSES

Numerical Linear Algebra	2 hours/week
Mathematical Programming	2 hours/week
Combinatorix (Graph Theory)	2 hours/week
Differential Equations	2 hours/week
Mathematical Programming	2 hours/week

CONTENTS

	Page
Summary	1
Chapter I : General Concepts of Initial-Boundary-Value Problems For Partial Differential Equations.	
1.1 Introduction.	1
1.2 First order equations.	3
1.3 Types of second-order Equation.	5
1.4 Initial and boundary conditions.	6
1.5 Existence of solutions.	8
1.6 Uniqueness of solution.	12
1.6.1 The uniqueness of solution for the IBVP of parabolic type.	13
1.6.2 The uniqueness of solution for the IBVP of Hyperbolic type.	17
1.7 Well-Posed and ill-posed Problems.	19
1.7.1 Examples for ill-posed problems.	20
1.7.2 Cauchy problem for the Heat equation as an example for a well posed problem.	23
1.8 Systems and Single Equations.	31
Chapter II : Finite Difference Schemes Formulation and Properties.	
2.1 Finite Difference Grid.	36
2.2 The Finite Difference Approximation to Derivatives.	37
2.3 Basic Difference Operators.	43

2.4	Basic Concepts of the Finite Difference Method Via the Heat Equation.	47
2.4.1	Formulation of Finite Difference Equations.	48
2.4.2	Errors Associated with Finite Difference Technique.	51
2.5	Further Methods for Obtaining Finite Difference Equations.	59
2.5.1	The Integral Method.	59
2.5.2	Polynomial Fitting.	61
Chapter III	: General Approach for Convergence of Finite Difference Approximations.	
3.1	The Function Space of an Initial-value Problem.	64
3.2	Fourier Series Approach.	65
3.3	Finite Difference Equations.	67
3.4	Lax's Equivalence Theorem.	71
3.5	Consistency of the Difference Equations with the Differential Equations.	74
Chapter IV	: The Stability Analysis (Methods and Examples).	
4.1	Introduction.	76
4.2	Survey of The Difference Equations Imposed for the Heat Equation.	76
4.3	The Matrix Method.	82

4.4	von Neumann Method.	88
4.5	The Energy Method.	91
4.6	The Wave Equation.	96
4.6.1	An Explicit Finite Difference Method.	96
4.6.2	Conversion to Two First Order PDE's.	99
Chapter V	: Matrix Stability and New Divided Differences Formulae.	
5.1	Introduction.	101
5.2	The Kreiss Theorem.	101
5.3	Observations on the divided Differences Via Leibnitz rule.	109
References.		123
Arabic Summary.		

SUMMARY

SUMMARY

The finite difference method for solving Initial-Boundary-value Problems for Partial Differential Equations is an approximate method, in the sense that derivatives at a point are approximated by difference quotients over a small interval. Accordingly, this method reduces the problem to difference equations forming a system of algebraic equations of the solution at different points.

In chapter I we introduced some advanced concepts from the theory of partial differential equations such as existence, uniqueness of solutions, well-posed and ill-posed problems and the classification of the equations and the associated initial and boundary conditions, with physical examples.

In Chapter II the concepts of the finite difference method is introduced, and we have considered three methods for developing difference equations, Taylor-series method, the integral method and the method of polynomial fitting. Considering the associated discretization and rounding errors, the convergence, stability and consistency.

In chapter III it is studied the theory of finite difference approximation in Banach space by using the discrete Fourier analysis also it is considered the prove of the Lax Equivalence theorem imposed for the initial-boundary-value problems.

In chapter IV we have introduced a survey of the partial difference equations imposed for the heat and wave equations also we considered the representation of the wave equation as a system of two first order equations. For studying the stability of a difference scheme, it is considered the Von Neumann method, the Matrix method and the Energy method.

In chapter V we have studied some matrix theorems, which we have recognized during studying the stability of difference equations. Also, in this chapter we have introduced some new formulas for the divided differences of the product of functions, and we have considered the analogy between the divided differences and the derivatives which introduced as a conjecture by Morton [41] and we published this work in march, 1989 [45].

CHAPTER I

**GENERAL CONCEPTS OF
INITIAL BOUNDARY VALUE PROBLEMS
FOR PARTIAL DIFFERENTIAL EQUATIONS**

CHAPTER I

GENERAL CONCEPTS OF INITIAL BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

1.1 Introduction:

Partial Differential Equations are those which contain one or more partial derivatives. They must, therefore, involve at least two independent variables. The order of a partial differential equation is the order of the highest derivatives, it is assumed finite.

A partial differential equation, abbreviated PDE, for one unknown function u of two independent variables x and y is a relationship

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, \dots) = 0 \quad (1.1)$$

where

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \text{ etc;}$$

Usually we use the notations:

$$u_x = p, \quad u_y = q, \quad u_{xx} = r, \quad u_{xy} = s, \quad u_{yy} = t \quad (1.2)$$

A linear equation is one which is linear in the dependent variable u and all of its derivatives occurring in the equation. Then, a linear equation can be written in the form

$$L u = g(x, y) \quad (1.3)$$

where $L u$ is a sum of terms each of which is a product of a function of x and y with u or one of its partial derivatives.

For first and second-order equations the respective forms for Lu are.

$$Lu = a u_x + b u_y - c u \quad (1.4)$$

and

$$Lu = a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u \quad (1.5)$$

where a, b, c, d, e and f are functions in x and y .

A linear equation is said to be homogeneous when, in equation (1.3) $g(x,y) = 0$, and non-homogeneous when $g(x,y) \neq 0$. Of course, a single partial differential equation for one unknown function, such as,

$$Lu = g(x, y) \quad (1.6)$$

where L is defined by formula (1.5), which also can be written in the form

$$F(x, y, p, q, r, s, t) = 0 \quad (1.7)$$

might possess many solutions, and one of the first questions to arise concerns their multiplicity and the auxiliary data that might serve to distinguish them from one another in a unique way.

The definition of L shows that the difference between two solutions of equation (1.3) is a solution of the corresponding homogeneous equation. Thus, the solutions of the inhomogeneous equation can be obtained by adding a solution of the homogeneous equation to any particular solution of the inhomogeneous equation this is analogous to that in ordinary differential equations.

One of the most important differences between the solutions of PDE's and those of ordinary differential equations. For whereas

the general solution of a linear ordinary differential equation contains arbitrary constants of integration, the general solution of a linear partial differential equation contain arbitrary functions. Garabedian [16], pointed that the general solution of a partial differential equation of order n for a function of k independent variables ought to depend on n arbitrary functions of $(k-1)$ independent variables, Cauchy-Kowalewski theorem.

Non-linear and Quasilinear equations, when the coefficients of an n^{th} order PDE depend upon n^{th} order derivatives, the equation is non-linear, when they depend upon m^{th} order derivatives, $m < n$, the equation is quasilinear Lapidus [32], Williams [61].

1.2 First-Order Quasilinear Partial Differential Equations:

The first order quasilinear PDE, in two independent variables x and y , can be written, Moon [40], as follows:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad (1.8)$$

The PDE, (1.3) can be considered as a special case of (1.8). From the geometric point of view, Lapidus [32] and Garabedian [16], the solution function $u(x, y)$ for equation (1.8) represents a surface $u = u(x, y)$ in the three dimensional space (x, y, u) , at each point of the solution surface the vector $(u_x, u_y, -1)$ has the direction of the normal to the surface. Thus, from equation (1.8) the vector (a, b, c) is perpendicular to the normal, and therefore must lie in the tangent plane of the surface $u = u(x, y)$. Accordingly the PDE is a mathematical statement of the geometrical requirement that any solution surface through a point $P(x, y, u)$ must be tangent

to a vector with components (a, b, c) . Furthermore, since (a, b, c) is always tangent to the surface. We never leave the surface. Note also that since $u = u(x, y)$ we have

$$du = u_x dx + u_y dy \quad (1.9)$$

and thus $(a, b, c) \parallel (dx, dy, du)$ and the general solution of equation (1.8) is given, Courant [9], by

$$G(v, w) = 0 \quad (1.10)$$

where G is an arbitrary function and where

$$v(x, y, u) = C_1 \quad \text{and} \quad w(x, y, u) = C_2$$

form a solution of the equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \quad (1.11)$$

Equation (1.11) comprises a set of two independent ordinary differential equations, a two-parameter family of curves in space, one of them,

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)}$$

is the characteristic curve.

Note that:

When $a = a(x, y)$ and $b = b(x, y)$, the characteristic is known as a base curve. When a and b are constants, the last equation defines a set of parallel lines in (x, y) space, in the general quasilinear case

$$\text{i.e. } a = a(x, y, u) \quad \text{and} \quad b = b(x, y, u)$$

the last equation can not be evaluated until $u(x, y)$ is also known.