CHARACTERISTIC INDICES AND LIAPUNOV'S FUNCTIONS FOR FAMILIES OF LINEAR DIFFERENTIAL SYSTEMS

THESIS

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SUMMARY

SUMMARY

One of the fundamental ideas in the qualitative theory of differential equations is the stability of solutions of a given differential systems. A. M. Liapunov [1892] gave the definitions of the stability, asymptotic stability of solutions.

This thesis consists of three chapters. The first contains a necessary background on stability theory, which is established by A. M. Liapunov [1892]. The solution $x(t)$ of the vector differential equation $\dot{x} = f(t,x)$ is said to be stable under an initial condition and external forces on the interval $[T, \infty)$, if it has small perturbation corresponding to small enough perturbation in the initial conditions or the external forces.

In the second chapter we expose the two main methods for studying the stability theory :

1] Method of characteristic numbers: the characteristic number (index) of a vector function $x(t)$ is defined by

$$\lambda_x = \overline{\lim_{t \to \infty}} \frac{1}{t} \ln \| x(t) \| .$$

These numbers are used to discuss the stability of the solutions of the linear differential system $\dot{x} = A(t)x$.

2] Method of Liapunov functions: this method depends on getting a scalar function $v(t,x)$; $x \in \mathbb{R}^n$ with special features to discuss the stability of a solution of differential system $\dot{x}=f(t,x)$, where $f\colon \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

The necessary results concerning these methods are mentioned in this chapter.

It is remarkable to notice that there is no general techinique to construct Liapunov functions and in most applications, in mechanics and physics, where it is necessary to study the stability of solution, these functions are obtained by guessing.

In chapter 3 we study general properties of the class of Liapunov functions, denoted by L_f, corresponding to a given vector differential equation $\dot{x}=f(t,x)$ with stable zero solution and also the general properties of a class of vector differential equations, denoted by S_v, having the same Liapunov function $v(t,x)$. And we found that these general properties help us in constructing the Liapunov function for certain types of differential equations.

In this chapter we, also, discuss the stability of the solutions of a given second order linear differential equation

$$\frac{d^2Y}{dt^2} + \theta a_1(t)\frac{dY}{dt} + a_2(t)Y = 0$$

where $\theta = \pm 1$, and $a_1(t)$, $a_2(t)$ are arbitrary bounded positive functions which guarantee the existence and uniquence of solutions, by constructing Liapunov function in the form of a quadratic form with constant coefficients. The obtained results inables us to get up upper bound of the characteristic indices of these equations.

CHAPTER I

STABILITY THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

The stability theory is that area of mathematics that deals with the problem of investigating the conditions that will not allow the solutions of a vector differential equation to deviate remarkably when sufficiently small changes in the initial conditions is made. F. M. Liapunov [1892] made great effort for introducing the fundamental definitions and theorems of this branch.

§.1.1. Basic notions and definitions

Let J be an interval of the real line $\mathbb{R}$ and $D \subset E^n$ be an open connected subset of the n-dimensional Euclidean space E^n with usual norm

$$\| x \| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} , \; x \equiv (x_1, \; \dots, \; x_2) \in E^n .$$

Consider

$$f: J \times D \longrightarrow E^n \; , (t, \; x) \longrightarrow f(t,x)$$

be a vector valued function defined on J×D.It is easy to see that f defines a scalar functions f_i: $J \times D \longrightarrow \mathbb{R}$, $i = 1$, ..., n , by

$$[f(t,x_1,\dots,x_n)]_i = f_i(t,x_1,\dots,x_n) \; .$$

Definition (1.1.1)

A differentiable vector valued function $\psi: J_1 \longrightarrow D$, $J_1 \subseteq J$, which satisfy $\dot{\psi}(t)=f(t, \psi(t))$ for every $t \in J_1$ is called a solution on J_1 of the vector differential equation

$$\dot{x} = f(t, x) \qquad , \qquad (1.1.1)$$

where "$\cdot$" is the derivative with respect to t.

In case when there is no such function, we say that the vector differential equation (1.1.1) has no solution.

Definition (1.1.2)

Let (t_o, x_o) be an arbitrary point in $J \times D$ and $J_o \subseteq J$ be an interval containing t_o. The solution $\psi(t)$ of the equation (1.1.1) on J_o which satisfies $\psi(t_o) = x_o$ is called a solution of the initial value problem

$$\dot{x} = f(t, x) \ , \ x(t_o) = x_o \qquad (1.1.2)$$

Definition (1.1.3)

A vector valued function $f: J \times D \longrightarrow E^n$, $(t, x) \longmapsto f(t, x)$, is called Lipschitzian in x uniformly with respect to t if there exists a constant $M > 0$ such that for every $(t, x), (t, y) \in J \times D$:

$$\| f(t,x) - f(t,y) \| \le M \| x - y \| .$$

Definition (1.1.4)

A vector valued function $f: J \times D \longrightarrow E^n$, $(t, x) \longmapsto f(t, x)$, is called piece-wise continuous function on J if for every fixed $x \in D$, the vector valued function $g_x: J \longrightarrow E^n$, $g_x(t) = f(t,x)$ is piece- wise continuous on J.

We denote by $C^p_{Lip}(J \times D)$ the class of all vector valued functions $f: J \times D \longrightarrow E^n$ which are piece-wise continuous on J and Lipschitzian in x uniformly with respect to t.

Through our thesis we assumed that the vector valued function $f: J \times D \longrightarrow E^n$ belongs to the class $C^p_{Lip}(J \times D)$. This assumption guarantees the existence and uniqueness of the solution for the initial value problem (1.1.2) on some interval $J_o \subseteq J$.

Let $f \in C^p_{Lip}(J \times D)$, $x(t)$ be a solution of the equation (1.1.1)on an interval $\Delta \equiv [a,b]$ and let $\delta > 0$. Let the region $U_x(\delta)$ be defined by

$$U_x(\delta) = \left\{ (t_o, x_o) \; : \; t_o \in \Delta, \; \| x_o - x(t_o) \| < \delta \right\} .$$

The following theorem gives more important properties of the solutions of initial value problem (1.1.2).

Theorem (1.1.1)

Let $f \in C^p_{Lip}(J \times D)$ and $x(t)$ be a solution of the vector differential equation $\dot{x} = f(t, x)$ on Δ. Then there exists $\delta > 0$ such that for every $(t_o, x_o) \in U_x(\delta)$, there exists a unique solution $x_{t_o, x_o}(t)$ of the initial value problem (1.1.2) passing through

(t_o, x_o). Moreover, the vector valued function

$$\psi : J\times U_X(\delta) \longrightarrow D,\ \psi(t, t_o, x_o) = x_{t_o, x_o}(t)\ ,$$

is continuous solution of the equation (1.1.2).

REMARKS

1- The function ψ: $J\times U_X(\delta) \longrightarrow D$ in the above theorem is well defined, this can be seen as follows :

(a) For a fixed point $(t_o, x_o) \in U_X(\delta)$, define a function

$$\bar{\psi}: J\times\{(t_o, x_o)\} \longrightarrow D\ ,$$

by $\bar{\psi}(t,t_o,x_o) = \chi_{t_o,x_o}(t)$. By the uniqueness of the solution of the equation (1.1.2), this function is well defined.

(b) Since the set $J\times U_X(\delta)$ can be written as a union of mutually disjoint sets $J\times\{(t_o, x_o)\}, (t_o, x_o)\in U_X(\delta)$, then we can define the function $\psi(t, t_o, x_o)$ by

$$\psi(t, t_o, x_o) = \bar{\psi}(t, t_o, x_o).$$

2- The continuity of the function $\psi(t, t_o, x_o)$ in the initial points (t_o, x_o) is called continuously dependence on the initial conditions.

§.1.2. Stability Theory

Let $I=[\tau, \infty]$ be an interval of $\mathbb{R}$, $\tau \geq 0$ and $B_\rho=\left\{x\in E^n : \|x\|<\rho\right\}$

be an open ball of radius ρ centered at the origin. Consider a vector differential equation

$$\dot{x} = f(t, x), \tag{1.2.1}$$

Where $f \in C^{p}_{Lip}(I \times B_{\rho})$. It is clear that the properties of the vector valued function $f(t, x)$ guarantee the existence and uniqueness of the solutions $\psi(t) \equiv \psi(t, t_o, x_o)$ of the initial value problem (1.2.1) and $x(t_0) = x_o$, $t_o \in I$, $x_o \in B_{\rho}$.

We assume always that the vector valued function $f: I \times B_{\rho} \longrightarrow E^{n}$ belongs to the class $C^{p}_{Lip}(I \times B_{\rho})$.

A. M. Liapunov [1892] introduced the following definitions of the stability of a solution of the equation (1.2.1)

Definition (1.2.1)

A solution ψ of the equation (1.2.1) is called stable at t_o ; $t_o \in I$, (or stable with respect to the initial conditions) if for every $\varepsilon > 0$, there exists $\delta \equiv \delta(t_o, \varepsilon) > 0$ such that if $\bar{\psi}$ is any other solution of (1.2.1) with $\| \bar{\psi}(t_o) - \psi(t_o) \| < \delta$, then $\| \bar{\psi}(t) - \psi(t) \| < \varepsilon$ for all $t \ge t_o$. Otherwise, ψ is called unstable at t_o.

Definition (1.2.2)

A solution ψ of the equation (1.2.1) is called asymptotically stable if it is stable and there exists a $\delta_1 > 0$ such that for any other solution $\bar{\psi}(t)$ of (1.2.1), $\| \bar{\psi}(t_o) - \psi(t_o) \| < \delta_1$ implies

$$\| \bar{\psi}(t) - \psi(t) \| \longrightarrow 0 \text{ as } t \longrightarrow \infty .$$

Remarks

Studying the stability of some solution ψ of the vector differential equation (1.2.1) may be reduced to studying the stability of a zero (trivial) solution. Because if we replace the variable x by a new variable $y \equiv x-\psi$, then the equation (1.2.1) becomes

$$\dot{y} = g(t, y) \qquad (1.2.2)$$

where $g(t, y) = f(t, y+\psi) - f(t, \psi)$. It is clear that $g(t, 0) = 0$, for all t, and hence $Y = 0$ is a solution of the vector differential equation (1.2.2). Thus the stability of the zero solution of equation (1.2.2) is equivalent to the stability of the solution $\psi(t)$ of the equation (1.2.1). Thus, without loss of generality, we assume that $f(t, 0) = 0$ for every $t \in I$, so that the equation (1.2.1) has always the zero solution. Consequently, for the zero solution the definitions of the stability can be formulated as follows.

Definition (1.2.3)

The zero solution of the equation (1.2.1) is called stable if for every $\varepsilon>0$, and every $t_o \in I$, there exists a $\delta \equiv \delta(t_o, \varepsilon)>0$ such that for every $x_o \in B_\delta$, we have $\| \psi(t, t_o, x_o) \| < \varepsilon$, for every $t \geq t_o$.

Definition (1.2.4)

If the quantity δ in the definition (1..2.3) is independent of t_o, or more precisely, if for every $\varepsilon > 0$, there exists a $\delta \equiv \delta(\varepsilon) > 0$ such that for every $t_o \in I$ and for every $x_o \in B_\delta$, we have $\| \psi(t, t_o, x_o) \| < \varepsilon$, for every $t_o \geq t$, then the zero solution of the equation (1.2.1) is called uniformly stable.

Definition (1.2.5)

The zero solution of the equation (1.2.1) is called asymptotically stable if it is stable and if for every $t_o \in I$, there exists $\delta_1 \equiv \delta_1(t_o) > 0$ such that for all $x_o \in B_{\delta_1}$, $\psi(t, t_o, x_o)$ is defined for all $t \geq t_o$ and for every $\eta > 0$, there exists a $T \equiv T(t_o, x_o, \eta) > 0$ such that for every $t \geq t_o + T$, $\| \psi(t, t_o, x_o) \| < \eta$.

Definition (1.2.6)

The zero solution of the equation (1.2.1) is called uniformly asymptotically stable if it is uniformly stable and if there exists a $\delta_1 > 0$ such that for every $\eta > 0$, there exists a $T \equiv T(\eta) > 0$ such that for every $t_o \in I$ and for every $x_o \in B_{\delta_1}$, we have $\psi(t, t_o, x_o)$ is defined for all $t \geq t_o$ and $\| \psi(t, t_o, x_o) \| < \eta$, for all $t \geq t_o + T$.

Definition (1.2.7)

A vector differential equation (1.2.1) is called stable (or uniformly/asymptotically/uniformly stable) if all its solutions