ON THE K-COHOMOLOGY GROUPS

THESIS

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By

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وقل اعملوا فسيرى الله عملكم ورسوله والمؤمنون

«صدق الله العظيم»

بسم الله الرحمن الرحيم

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M. Sc. COURSES

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(i) Algebraic topology — 2 hours weekly for two semisters

(ii) Functional analysis — 2 hours weekly for two semisters.

(iii) Linear algebra — 2 hours weekly for two semisters

(iv) Logical mathematics and graph theory — 2 hours weekly for two semisters

(v) Modern algebra — 2 hours weekly for one semister

(vi) Higher differentional geometry — 2 hours weekly for one semister

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SUMMARY

PREFACE

The cohomology theory is one of the important subjects in algebraic topology, its axiomatic characterization has been given firstly by S. Eilenberg and N. steenrod, [11], as that for the homology theory. Its importance is due to the fact that an algebraic representation of topology converts topological problems into algebraic ones, to the end that with sufficiently many representations, the topological problems will be solvable if (and only if) all the corresponding algebraic problems are solvable.

In his paper [5] D.O. Baladze has introduced one of the most interesting generalizations of the steenrod's homology groups [18] . D.O. Baladze proposed the idea of the K-homology and K-cohomology groups for compact metric spaces. The definition and duality of the relative homology and cohomology K-groups of an arbitrary topological space over a pair of coefficient groups based on a partition system have been suggested by A. Dabbour, [1] .

In the present thesis we give a relation between the K-cohomology and K-homology groups of a compact (not metric) space over a pair of coefficient groups based on a system of open coverings of the space and the corresponding groups based on a system of partitions. We study the duality of these groups. Also we discuss the K-cohomology construction from the point of view of the first four axioms of the Eilenberg - steenrod axioms. The thesis consists of three chapters.

The first chapter contains basic ideas from algebra, topology and algebraic topology. These ideas form the theoretical base of our study.

In the first article of the second chapter the construction of the K-homology and K-cohomology groups of a pair of compact spaces over a pair of coefficient groups relative to a system of open coverings are introduced.

In the second article we define the K-homology and K-cohomology groups of compact spaces relative to a system of partitions. Also the relation between these groups and those of the first article of this chapter is discussed.

In order to make the groups defined in the first article important we discuss the K-homology and K-cohomology sequences corresponding to a short exact sequence of coefficient groups in the first article of the third chapter. In the second article of the third chapter we prove that the K-cohomology constnuction satisfies the first four axioms of the seven axioms of Elenberg-steenrod. The third article of the second chapter and the third chapter seem to be important and original.

CHAPTER ONE

CHAPTER I

SOME BASIC CONCEPTS AND CONVENTIONS

In this chapter we introduce the definitions and the conventions which are required for our study in this thesis. We clasify them into five main sections. In the first section we consider a general topological and algebraic concepts. In the second section we introduce the idea of the categories, functors and cohomology theory. The third section deals with the inverse and direct systems of groups and their limits In the fourth section we elaborate chain (cochain) complexes, simplicial complexes and their chain (cochain) complexes which are leading to the construction of their homology (cohomology) groups. Finally we concern with the duality theory in the last section, and also in this section we consider the direct limit of a direct system of compact groups.

I.1. General Topological And Algebraic Concepts.

In this section we introduce some general topological and algebraic preliminaries which will be used later. All groups under consideration in our thesis are commutative groups.

Definition I.1.1

Let $\{X_\alpha\}$ be a collection of sets indexed by a set M, i.e. for each $\alpha \in M$, X_α is a set of the collection . The product $\prod_{\alpha \in M} X_\alpha$ of the collection $\{X_\alpha\}$ is the totlity of functions $x = \{x_\alpha\}$ defined for each $\alpha \in M$ and such that x_α , the value of x on α, is an element of X_α . The element x_α is called the α-coordinates of x.

In case the sets X_α all coincide with a set X, then the product $\prod_{\alpha \in M} X_\alpha$ is denoted by X^M, and is simply the set of all functions from M to X. If $M = \{1, 2, \ldots, n\}$ then the product $\prod_{\alpha \in M} X_\alpha$ is written $X_1 \times X_2 \times \ldots \times X_n$.

If each X_α is a topological space, a topology is introduced in the product of the collection $\{X_\alpha\}$ as follows:

If a finite number of X_α ,s are replaced by open subsets $U_\alpha \subset X_\alpha$, the product of the resulting collection is a subset of $\prod_{\alpha \in M} X_\alpha$, and is called a rectangular open set of $\prod_{\alpha \in M} X_\alpha$. Any union of rectangular open sets is called an open set of the product.
The product with this topology is called the topological product, and the space X_α is called α-component of the product space, [12] , [20] .

If each X_α is a group, then an addition is defined in $\prod_{\alpha \in M} X_\alpha$ by the usual method of adding functional values:-

$$(x+x')_\alpha = (x+x')(\alpha) = x(\alpha) + x'(\alpha) = x_\alpha + x'_\alpha$$

In this way, the product $\prod_{\alpha \in M} X_\alpha$ becomes a group and is called the direct product of the groups $\{X_\alpha\}$, [12] .

Definition I.1.2.

Let $\{G_\alpha\}$ be an indexed collection of groups by a set M, their direct sum $\sum_{\alpha \in M} G_\alpha$ is the subgroup of their direct product $\prod_{\alpha \in M} G_\alpha$ consisting of those elements having all but a finite number of coordinates equal to zero, i.e., $g_\alpha = 0 \in G_\alpha$ for all but a finite number of $\alpha \in M$. In case M is a finite set, then of course the direct sum and direct product coincide , [12] .

Definition I.1.3.

A topological pair (X,A) consists of a topological space X, and a subspace A. If $A = \emptyset$ we usually do not distinguish between the pair $(X,\emptyset)$ and the space X.

Definition I.1.4.

The lattice of a topological pair (X,A) consists of the pairs:-

$$(\emptyset,\emptyset) \longrightarrow (A,\emptyset) \begin{array}{c} \nearrow (X,\emptyset) \searrow \\ \searrow (A,A) \nearrow \end{array} (X,A) \longrightarrow (X,X) ,$$

all their identity maps, the inclusion maps indicated by arrows, and all their compositions. If $f: (X,A) \rightarrow (Y,B)$ is a continuous map, then f defines a continuous map of every pair of the lattice of (X,A) into the corresponding pair of the lattice of (Y,B), [12] .

Definition I.1.5.

The two maps $f_o, f_1: (X,A) \longrightarrow (Y,B)$ are said to be homotopic if there is a continuous map

$$h : (X,A) \times I \longrightarrow (Y,B)$$

where I denotes the closed unit interval

$I = [0,1]$ such that $f_o(x) = h(x,0)$, $f_1(x) = h(x,1)$; for each element $x \in X$. The map h is called a homotopy, [17].

Definition I.1.6.

A partition D_α of a space X is a collection $\{e_r^\alpha\}$ of subsets of X indexed by a set I such that:

$$\bigcup_{r \in I} e_r^\alpha = X \quad \text{and} \quad e_r^\alpha \cap e_{r'}^\alpha = \emptyset \quad \text{when } r \neq r',$$

If I is a finite set, then D_α is called finite partition [9].

Definition I.1.7.

An indexed family $O = \{O_i\}_{i \in M}$, of open sets in a topological space X, with the set M of indices, is called an open cover, or open covering, of X if, $X = \bigcup_{i \in M} O_i$. If A is a subset of X, M' is a subset of M such that $A \subset \bigcup_{j \in M'} O_j$, then the pair (O, O^A), $O^A = \{O_j\}_{j \in M'}$, is said to be an open covering of the pair (X,A).

Let $O = \{O_i\}_{i \in M_1}$ and $O' = \{O'_j\}_{j \in M_2}$ be two coverings of the space X. The covering O' is called a refinement of the covering O (notation : $O < O'$) if every member of O' is contained in some member of O, [20].

Definition I.1.8.

A space X is called compact if every open cover of X has a finite subcover. A pair (X,A) is called compact if X is compact and A is closed (and therefore compact) subset of X.

For the compact spaces we have the following theorem [20].