Ain Shams University
Faculty of Science
Mathematics Department

Some Advanced Valuations of Graphs

A thesis
submitted for the award of the Ph.D. degree
in Pure Mathematics

By

Manal Naji Yaqoob

(*M.Sc.Degree in Applied Mathematics, University of Technology, Baghdad, Iraq*)

Supervised By

Professor Mohammed Abd-El Azim Seoud

Professor of Pure Mathematics
Mathematics Department
Faculty of Science
Ain Shams University

Dr.Alaa El-Din Ismael Abd-ElMaqsoud

Lecturer of Pure Mathematics
Mathematics Department
Faculty of Science
Ain Shams University

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Finally, I would like to thank each members of *Mathematics Department* who helped in bringing out my thesis to life.

Manal

2014

Dedication

To
The memory of my late father,
my beloved husband and my sons
Hasan and Husain
For their patience and support

Summary

In this thesis we are concerned with a very important topic in graph theory, which is graph labeling. We study seven labelings of graphs, combination labeling, divisor labeling, divisor cordial labeling, $*$-divisor labeling, square sum labeling, perfect square sum labeling, and strongly square sum labeling. This thesis consists of four chapters.

Chapter one

In Chapter one, we introduce some basic definitions and notations in graph theory which we will need afterwards. Also, we deal with preliminary definitions and results about the related work in this thesis which have been done before.

Chapter two

In Chapter two, we study combination graphs. We introduce some theorems for a graph to be a non-combination graph, and some theorems on chains of two and three complete graphs, considering when they are combination or non-combination graphs. Also we present some families of combination graphs. Finally we give a survey for trees of order n; $n \leq 10$, which are all combination graphs.

Chapter three

In Chapter three, we discuss three kinds of divisor graphs, namely the usual divisor graph, the divisor cordial graph and a new divisor graph called $*$ - divisor graph.

Chapter four

In Chapter four, we introduce a type of labeling of graphs which is closely related to the Diophantine Equation $x^2+y^2 = z$. We introduce some families to be square sum graphs. We determine the number of strongly square sum graphs corresponding to the number of edges. We

present a program finding the maximum number of edges of square sum and perfect square sum graphs.

List of publications arising from this thesis

[1] M. A. Seoud , M.N. Al-Harere, *Some Notes on Combination Graphs*, accepted for publication in Ars Combinatoria.

[2] M.A.Seoud, M.N. Al-Harere ,*On Combination Graphs*, Int.Math.Forum ,7(2012)1767-1776.

[3] M. A. Seoud , M.N. Al-Harere, *Some Non-Combination Graphs*, Applied Mathematical Sciences, 6(2012), 6515 - 6520 .

[4] M. A. Seoud , M.N. Al-Harere, *Three Kinds of Divisor Graphs*, preprint.

[5] M. A. Seoud , M.N. Al-Harere, *Further Results on Square Sum Graphs*, accepted for National Academy Science Letters, Springer.

Introduction

The existence of graphs for which a special set of integer values are assigned to its nodes or edges or both according to some given criteria has been investigated since the middle of the last century. Graph labeling have often been motivated by practical considerations such as chemical isomers, but they are also of interest in their own right due to their abstract mathematical properties arising from various structural considerations of the underlying graphs.

The qualitative labeling of graph elements have inspired research in diverse field of human enquiry such as conflict resolution in social psychology, electrical circuit theory and energy crisis. Whereas quantitative labeling of the graphs have led to quite intricate fields of applications such as coding theory problems, including the design of good radar location codes, synch-set codes, missile guidance codes and convolution codes with optimal auto correlation properties. Labeled graph often been applied in determining the ambiguities in X-ray crystallographic analysis, to design communication networks, in determining optimal circuit layouts and radio astronomy etc. See [4],[5],[22] for details.

Owing to its wide applications in diverse fields of knowledge, an enormous body of literature has grown around the theme. More than 1500 papers on various graph labeling methods have been published in papers in the literature and a very complete dynamic survey by J. Gallian [7].

Besides their practical applications as indicated above, their theoretical applications too are numerous not only in the theory of graphs, but also in other areas of mathematics such as Combinatorial Number Theory, Linear Algebra and Group Theory admitting a given type of labeling [7]. Labeled graphs are also used to construct the polygons of same internal angles and distinct sides.

Interest in graph labeling problems began in the mid 1960s. A graph G consists of a set of vertices and a set of edges. Every edge must join two distinct vertices and no more than one edge may join any vertex pair.

If a non negative integer $f(v)$ is assigned to each vertex v of G then the vertices of G are said to be labeled (numbered). G is itself a labeled graph if each edge $e = uv$ is given the value $f(uv) = f(u) * f(v)$, where * is a binary operation. In the literature one can find that the * is either addition, multiplication, modulo addition or absolute difference, modulo subtraction or symmetric difference. Clearly, in the absence of additional constraints, every graph can be labeled in infinitely many ways. Thus, utilization of labeled graph models requires imposition of additional constraints which characterize the problem being investigated.
"Graph labeling" at its heart, is a strong communication between "number theory" and "structure of graphs".

Most graph labeling methods trace their origin to one introduced by Rosa [17] in 1967, or one given by Graham and Sloane [9] in 1980. Rosa [17] called a function f a β-valuation of a graph G with q edges if f is an injective function from the vertices of G to the set $\{0, 1, \dots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [8] subsequently called such a labeling graceful and this is now the popular term. Rosa introduced β-valuation as well as a number of other labeling as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [16] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges.

Graham and Sloane [11] introduced harmonious graphs. They defined a graph G with q edges to be harmonious if there is an injective function f from the set of vertices of G to the group of integers modulo q such that when each edge xy is assigned to the label $[f(x) + f(y)](mod q)$, the resulting labels are distinct. They prove that if a harmonious G has an even size q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Liu and Zhang [15] proved that every graph is a subgraph of a harmonious graph. Determining whether a graph has harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001[14].

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Contents

Chapter 1

Basic Definitions and Concepts

This chapter consists of two sections. Section one gives some definitions and notations in graph theory. Section two contains the preliminary definitions and results about the related work in this thesis which have been done before.

1.1 *Basic definitions in graph theory*

A *graph* G is defined by an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty set whose elements are called points or vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of the graph G. A graph G with n vertices and q edges is called a (n, q) - graph.

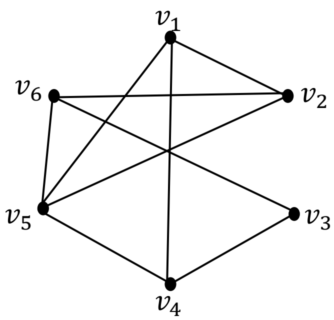

Figure 1.1: (6,9)−graph

The number of vertices in a given graph is called *order* of the graph, denoted by $|V(G)|$.The number of edges in a given graph is called *size* of the graph, denoted by $|E(G)|$.

An edge of a graph that joins a vertex to itself is called a *loop*. As an example, see *Figure* 1.2 (e_1).

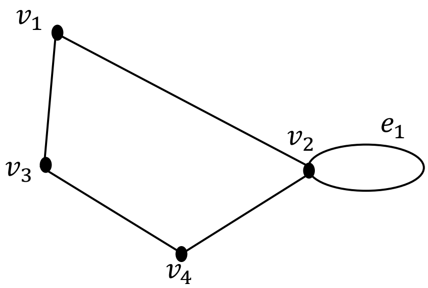

Figure 1.2: Graph with loop

If two vertices of a graph are joined by more than one edge then these edges are called *multiple edges. For instance, see Figure 1.3(* e_1 *and* e_2).

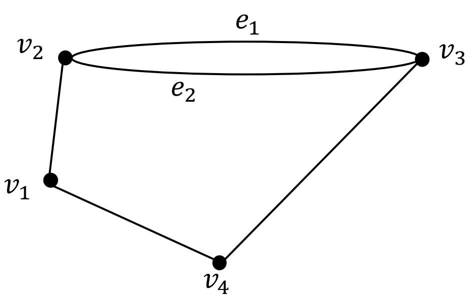

Figure 1.3: Graph with multiple edges

A graph which has neither loops nor multiple edges is called a *simple graph*. A graph is *finite* if both its vertex set and edge set are finite.

If two vertices of a graph are joined by an edge then these vertices are called *adjacent* vertices.

If two or more edges of a graph have a common vertex then these edges are called *adjacent edges*.

Degree of a vertex v of any graph G is defined as the number of edges incident on v. It is denoted by $deg(v)$ or $d(v)$.

A vertex of degree 0 is an *isolated vertex* and a vertex of degree 1 is an *end-vertex, leaf,* or a *pendant vertex.*

A graph H is a *subgraph* of G if every vertex of H is a vertex of G, and every edges of H is an edge of G. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that a subgraph H is a *spanning subgraph* of G if H contains all vertices of G.

A subgraph H of G, such that whenever $u, v \in V(H)$ are adjacent in G then they are adjacent in H is called an *induced subgraph* of G. (*for example, see Figure 1.4*)

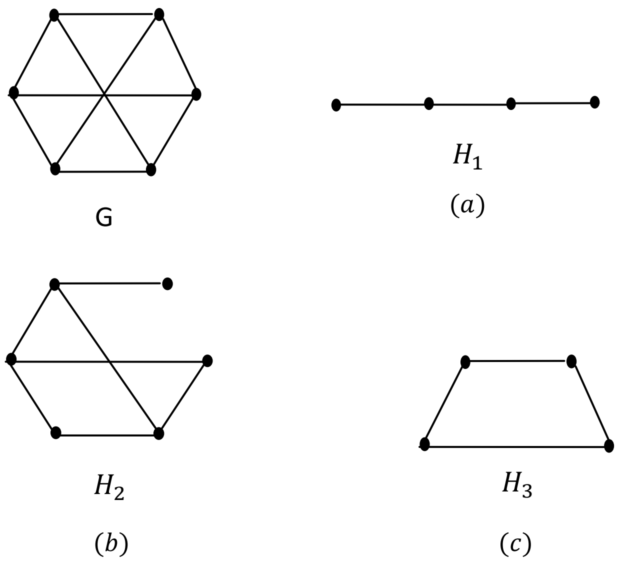

Figure 1.4:
(a) H_1 is a subgraph of G.
(b) H_2 is a spaning subgraph of G.
(c) H_3 is an induced subgraph of G.

The *complement* $\bar{G}$ of a simple graph G with vertex set $V(G)$ is the graph in which two vertices are adjacent if and only if they are not adjacent in G. (*for instance, see Figure 1.5*)

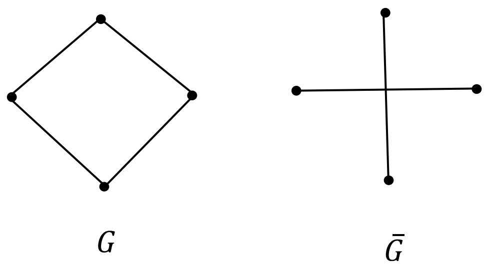

Figure1.5: Graph G with its complement

The two graphs G and H are *isomorphic* $G \cong H$ if there is a one to one correspondence ϕ mapping from $V(G)$ onto $V(H)$, such that ϕ preserves adjacency, i.e., for $u, v \in V(G), uv \in E(G)$ if and only if $\phi(u)\,\phi(v) \in E(H)$. Thus the two graphs shown in *Figure 1.6* are isomorphic under the correspondence $u \leftrightarrow l, v \leftrightarrow m, w \leftrightarrow n, x \leftrightarrow p, y \leftrightarrow q, z \leftrightarrow r$

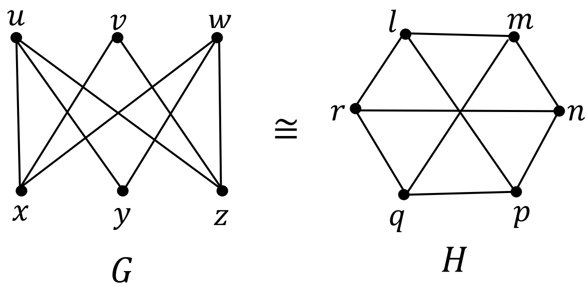

Figure 1.6: $G \cong H$

If e is an edge of a graph G, we denote by $G - e$ the graph obtained from G by deleting the edge e. Similarly, if v is a vertex of G, we denote by $G - v$ the graph obtained from G by deleting the vertex v together with the edges incident on v.

Figure 1.7 illustrates the deletion of vertex or edge from G

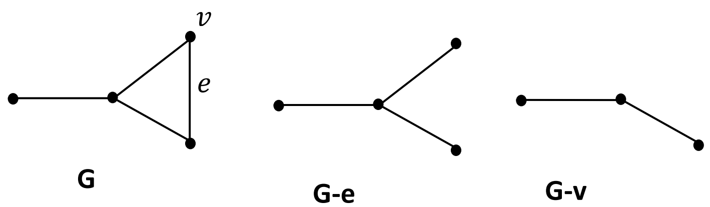

Figure 1.7

1.1.1 *Some important types of graphs*

A *regular graph* is defined as a graph that all of its vertices are of the same degree, in such case we say that the graph of degree d is a d-regular graph.

A *complete graph* of order n (K_n) is a regular graph of degree $n-1$ (*as an example, see Figure 1.8*).

A *null graph* is a regular graph of degree zero. Null graph of n vertices is denoted by $\overline{K_n}$. *(for example, see Figure 1.9).*

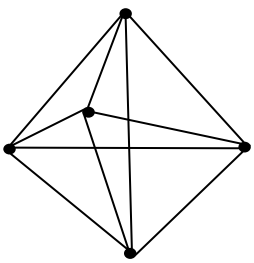

Figure 1.8: K_5

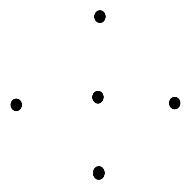

Figure 1.9: $\overline{K_5}$

If the vertices set of a graph G can be partitioned into two sets V_1 and V_2 such that any edge of G joins one vertex in V_1 to one vertex in V_2 then G is called a *bipartite graph* having bipartition (V_1, V_2).