

IONS

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P R E F A C E

The thesis deals mainly with infinite matrices associated with basic sets of polynomials. In most cases the matrices are algebraic ones.

The thesis consists of four chapters. Chapter I is an introduction to basic sets of polynomials and to algebraic infinite matrices. This introduction involves only those results which are needed in later chapters.

In chapter II we investigate effectiveness in closed circles, effectiveness in an open circle, effectiveness at the origin, and effectiveness in the whole plane for every entire function, of basic sets associated with "non-singular" matrix functions $F(P)$ where P is an algebraic row-finite matrix; four theorems, each concerning one of these four topics, are established. Applying these four theorems, another four theorems are proved on the effectiveness of basic sets associated with algebraic sum matrices. Two further theorems are given which deal with effectiveness of basic sets associated with non-singular matrix functions $F(P)$ where P is an algebraic row-finite matrix whose elements are of assigned magnitude.



In Chapter III, four theorems are established on the effectiveness of basic sets associated with matrix functions $\phi(P)$ when a basic set associated with some certain function $F(P)$ is known to be effective. A number of results on algebraic infinite matrices which are required for this purpose are given first. In this chapter too, two theorems are established concerning the effectiveness of basic sets associated with "upper row-finite matrices" which are not necessarily algebraic. Furthermore the effectiveness of basic sets associated with functions of algebraic upper row-finite matrices is investigated.

In chapter IV, we first investigate the order of basic sets associated with non-singular matrix functions $F(P)$ of an algebraic row-finite matrix P whose elements are of certain order of magnitude. We next investigate the order, as well as the type on a circle, of basic sets associated with non-singular matrix functions $F(P)$ of an algebraic semi-block matrix P . In the last article we investigate the order of basic sets associated with non-singular matrix functions $F(P)$ when the basic set associated with the algebraic lower semi-matrix (or the algebraic upper row-finite matrix) P is of given order. Theorems I and II of this chapter can be stated as theorems on algebraic

infinite matrices without referring to basic sets.

It is hoped that instalments of chapters II, III and IV would be published.

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CHAPTER I
1. INTRODUCTION TO BASIC SETS
OF POLYNOMIALS AND ALGEBRAIC INFINITE MATRICES

In this chapter we first give a very brief survey to the theory of basic sets of polynomials which we shall need in later chapters. No proofs are given since these results (and their proofs) are mostly contained in Whittaker's memoir [23]. We then give, in this chapter too, an account of some recent researches on algebraic infinite matrices which are in complete touch and are necessary to our work in later chapters. Proofs are included, since these results are gathered from diverse sources.

1. Basic Sets Of Polynomials.

A set of polynomials

$$(1) \quad \{p_n(z)\} = p_0(z), p_1(z), \dots, p_n(z), \dots$$

forms a basic set, if every polynomial can be expressed, in one and only one way, as a finite linear combination of the elements of the set. This definition implies that the elements of the set must be linearly independent.

Let the representations of the polynomials z^n , $n=0,1,2,\dots$ be

$$(2) \quad z^n = \sum_1 \prod_{n1} p_1(z), \quad n = 0,1,2,\dots$$

and let

$$(3) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be regular near the origin, then by (2) we have, formally,

$$(4) \quad f(z) = \Pi_0 f(0) p_0(z) + \Pi_1 f(0) p_1(z) + \Pi_2 f(0) p_2(z) + \dots$$

where

$$(5) \quad \begin{aligned} \Pi_n f(0) &= \Pi_{0n} a_0 + \Pi_{1n} a_1 + \Pi_{2n} a_2 + \dots \\ &= \Pi_{0n} f(0) + \Pi_{1n} f'(0) + \Pi_{2n} \frac{f''(0)}{2!} + \dots \end{aligned}$$

The series in (4) is called the associated basic series of $f(z)$.

Writing

$$(6) \quad p_i(z) = \sum_j p_{ij} z^j, \quad i = 0, 1, 2, \dots$$

then $P = [p_{ij}]$, $i, j = 0, 1, 2, \dots$ is called the matrix of coefficients of the set $\{p_n(z)\}$, whilst $\Pi = [\Pi_{ij}]$, $i, j = 0, 1, 2, \dots$ is called the matrix of operators associated with the set. Both P and Π are row-finite matrices. The following result has been proved by Whittaker:

Lemma 1: The necessary and sufficient condition for the set $\{p_n(z)\}$ to be basic is that the matrix of coefficients P should have a unique row-finite reciprocal Π .

Writing

$$(7) \quad \omega_n(R) = \sum_1 \left| \Pi_{n1} \right| A_1(R),$$

where

$$(8) \quad A_1(R) = \max_{|z|=R} |p_1(z)|,$$

Whittaker proved that when the function $f(z)$ in (3) is such that $\sum_{n=0}^{\infty} a_n |z|^n (R)$ converges, $f(z)$ is represented by its associated basic series in and on the circle $|z| = R$, by "representation" is meant the uniform convergence of the basic series to $f(z)$.

To distinguish between the different convergence properties of basic series, we have the following definitions:

- (i) A basic set is said to be effective in the circle $|z| < R$ if every function $f(z)$ regular in and on $|z| = R$ is represented by its associated basic series in and on $|z| = R$.
- (ii) A basic set is said to be effective in the circle $|z| < R$ when every function $f(z)$ regular in $|z| < R$ is represented by its associated basic series in $|z| < R$.
- (iii) A basic set is said to be effective at the origin when every function $f(z)$ regular at $z = 0$, is represented by its associated basic series in some circle surrounding the origin, the size of the circle being dependent on $f(z)$.
- (iv) A basic set is said to be effective at the origin (or in the whole plane) for every entire function when every entire function $f(z)$ is represented by its associated basic series in some circle surrounding the origin (or in the whole plane).

The number of non-zero coefficients $\prod_{n=1}^{\infty} a_n$ in the representation (2) is denoted by $N(n)$. A basic set satisfying

the condition

$$(9) \quad \{N(n)\}^{1/n} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty$$

is called a Cannon basic set. Writing

$$(10) \quad \lambda(R) = \overline{\lim}_{n \rightarrow \infty} \{\omega_n(R)\}^{1/n},$$

it has been proved, by Cannon, that if for a Cannon basic set $\lambda(R) > R$, for some value of $R > 0$, then there exists a function of radius of regularity ρ where $R < \rho < \lambda(R)$, which is not represented by the associated basic series, in $|z| \leq R$.

Combining this result with the former result due to Whittaker, we have the fundamental theorem on the effectiveness in a closed circle $|z| \leq R$, viz.

Lemma 2: The necessary and sufficient condition for a Cannon basic set to be effective in $|z| \leq R$ is that $\lambda(R) = R$.

In the case of general basic sets, i.e. sets not satisfying Cannon's condition (9), Cannon considers, in place of the expression $\omega_n(R)$ the expression

$$(11) \quad F_n(R) = \max_{i,j} \max_{|z|=R} \left| \prod_{n_1} p_1(z) + \prod_{n_{i+1}} p_{i+1}(z) + \dots + \prod_{n_j} p_j(z) \right|$$

Writing

$$(12) \quad \chi(R) = \overline{\lim}_{n \rightarrow \infty} \{F_n(R)\}^{1/n}$$

then, in place of lemma 2, we have

Lemma 2: The necessary and sufficient condition for a general basic set to be effective in $|z| \leq R$ is that $\chi(R) = R$

We note that in general

$$(13) \quad R \leq \chi(R) \leq \lambda(R)$$

but in the case of a Cannon set $\chi(R) = \lambda(R)$.

The fundamental theorems on the effectiveness in an open circle $|z| < R$, effectiveness at the origin, and effectiveness at the origin (or in the whole plane) for every entire function are as follows.

Lemma 3: The necessary and sufficient condition for a Cannon basic set to be effective in the circle $|z| < R$ is that

$$\lambda(r) < R \text{ for all } r < R.$$

Lemma 4: The necessary and sufficient condition for a Cannon basic set to be effective at the origin is that $\lambda(0+) = 0$.

Lemma 5: The necessary and sufficient condition for a Cannon basic set to be effective at the origin (in the whole plane) for every entire function is that $\lambda(0+) < \infty$ ($\lambda(R)$ is finite for all finite values of R).

For general basic sets, $\lambda(r)$ is replaced by $\chi(r)$ and $\lambda(0+)$ is replaced by $\chi(0+)$.

The properties of the functions $\lambda(R)$ and $\chi(R)$ were established mainly by Whittaker. For a Cannon set we have

$$(14) \quad \lambda(R^{1+a}) \leq \{\lambda(R)\}^{1-a/b} \{\lambda(R^{1+b})\}^{a/b}, \quad 0 < a < b,$$

and from (14) the following results have been deduced:

(1) There is at most one interval in which $\lambda(R)$ is a constant and moreover the left end point of this interval is zero.

(11) If $\lambda(R) = R$ for $R = a$ and $R = b$, ($0 < a < b$), then $\lambda(R) = R$ for all R such that $a \leq R \leq b$.

Similar results hold for $\chi(R)$, and so we have:

Lemma 6: The circles of effectiveness of a basic set form an annulus $a \leq R \leq b$ or $a \leq R < b$.

The number $D(n)$ has been introduced, by Cannon, as the degree of the polynomial of highest degree in the representation (2). It is clear that $N(n) \leq D(n) + 1$. Cannon has proved that if

$$(15) \quad \lim_{n \rightarrow \infty} \frac{D(n)}{n} = \alpha$$

then

$$(16) \quad \frac{\lambda(r)}{r^\alpha} \leq \frac{\lambda(R)}{R^\alpha}, \quad 0 < R \leq r$$

Thus for sets satisfying (15) which are consequently Cannon sets, $\lambda(R)$ is either finite and continuous for all $R > 0$ or infinite for all $R > 0$. Consequently we have

Lemma 7: If a Cannon basic set satisfying condition (15) is effective at the origin for every entire function, it is effective in the whole plane for every entire function.

For Cannon sets not satisfying (15), Nassif has shown that $\lambda(R)$ has at most one discontinuity beyond which it becomes infinite.

A simple set of polynomials is defined as one in which $p_n(z)$ is of degree n for all n . Such a set is basic and $D(n) = n$ for all n . The value $\alpha = 1$ in (16) gives Cannon's result:

Lemma 8: A simple set which is effective in some circle $|z| \leq R_1$ is effective in every circle $|z| \leq R$, $R \geq R_1$.

But the set may not be effective in a circle $|z| \leq R$, $R < R_1$.

We also have the result that a simple basic set which is effective in an open circle $|z| < R_1$ is also effective in the closed circle $|z| \leq R_1$ (and so effective in every circle $|z| \leq R$, $R \geq R_1$).

Whittaker has also considered the representation of entire functions, of definite order and type, by basic series of polynomials. Whittaker's fundamental theorem may be explained as follows.

The order ρ and type σ of an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{are defined by}$$

$$(17) \quad \rho = \overline{\lim}_{n \rightarrow \infty} n \log n / \log \frac{1}{|a_n|},$$

$$(18) \quad \sigma = \frac{1}{e^\rho} \left\{ \overline{\lim}_{n \rightarrow \infty} n^{1/\rho} |a_n|^{1/n} \right\}^\rho$$

provided $0 < \rho < \infty$.

Whittaker defines the order ω and type λ of a Cannon basic set as

$$(19) \quad \omega = \lim_{R \rightarrow \infty} \omega(R),$$

where

$$(20) \quad \omega(R) = \overline{\lim}_{n \rightarrow \infty} \log \omega_n(R) / n \log n,$$

and

$$(21) \quad \lambda = \lim_{R \rightarrow \infty} \lambda(R)$$

where

$$(22) \quad \lambda(R) = \frac{e}{\omega} \left[\overline{\lim}_{n \rightarrow \infty} \left\{ \omega_n(R) \right\}^{1/n} n^{-\omega} \right]^{1/\omega},$$

provided $0 < \omega < \infty$.

The fundamental theorem is now as follows:

Lemma 9: A Cannon basic set of order ω and type λ represents in the whole plane (is effective for) every entire function of increase less than order $1/\omega$, type $1/\lambda$, i.e., every entire function

(i) of order $< 1/\omega$,

(ii) of order $1/\omega$ but type $< 1/\lambda$.

Cannon has proved that the necessary and sufficient

condition for a Cannon basic set to be effective for every entire function of growth less than order p , type q is that

$$(23) \quad \overline{\lim}_{n \rightarrow \infty} \left[\left(\frac{p}{n} \right)^{1/p} \left\{ \omega_n(R) \right\}^{1/n} \right] \leq 1, \text{ for all } R.$$

Thus a Cannon basic set of order ω and type γ does not represent in the whole plane every entire function of order $p > 1/\omega$ or of order $p = 1/\omega$ but type $q > 1/\gamma$.

The function $\omega(R)$ may be finite and continuous for all R , but may increase indefinitely with R , in which case $\omega = \infty$. In such a case Ewida [2] calls $\omega(R)$ the order of the Cannon set $\{p_n(z)\}$ on the circle $|z| = R$, and proves that the basic set is effective in the circle $|z| \leq R$ for every entire function of order $< \frac{1}{\omega(R)}$.

In the case of a basic set satisfying the condition

$$(24) \quad \lim_{n \rightarrow \infty} \frac{D(n)}{n \log n} = 0,$$

and so the set is a Cannon set, $\omega(R)$ is the same for all $R > 0$, whether finite or infinite and this fixed value is then the order ω of the basic set [2].

When $\omega(R)$ is the same for all $R > 0$, $\gamma(R)$ may not be the same for all $R > 0$; it may increase to infinity with R . In such a case the basic set is effective in the circle $|z| \leq R$ for every entire function of order $\frac{1}{\omega}$, type $< \frac{1}{\gamma(R)}$,