ON STRONGLY RIGHT BOUNDED RINGS

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INTODUCTION

INTRODUCTION

In 1984, Faith and Page [21] defined a ring to be strongly right (left) bounded, SRB (SLB), if every nonzero right ideal contains a nonzero ideal. This definition is an extension of the definition of a bounded right ideal introduced earlier by Jacobson [28,P.38]. SRB rings are generalizations of right duo rings (that is, rings in which every right ideal is an ideal) and right bounded rings (that is, rings in which every essential right ideal contains a nonzero ideal). Recently, Birkenmeier, Heatherly, Tucci, and Xue have started studying SRB rings and SRB algebras [7], [8], [10], [11], and [40]. Also SRB rings play a fundamental role in the theory of right finitely pseudo - Frobenius (FPF) rings (that is, rings for which every finitely generated faithful module is a generator) [7] and [21]). From [21], every strongly right bounded right self injective ring is right FPF and the basic ring of a semiperfect right FPF ring is strongly right bounded. In 1987, Faith has conjectured that a right FPF ring is Morita equivalent to a SRB ring [16, P.310]. Weimin Xue has shown in a very recent work on SRB rings [40], that a finite duo ring with identity is left duo. He has shown the following two results:

<u>Theorem</u> [40,page 353] If R is strongly right

bounded with unity and the order of R has factors of the form p^5 (in particular $|R| < 32$), then R is strongly left bounded.

Proposition [40, Proposition 2] Let R be a strongly right bounded finite ring with unity and with p^4 elements, p is prime. If R is not local, then R is strongly left bounded.

A ring R is defined to be left Matlis ring [31] if for every injective indecomposable left R-module M there exists a prime ideal P of R such that $M \cong E(R/P)$, the injective hull of the left R-module R/P. The following result is a restatement of [31, Corollary 3.6] and it is basic to the study of injective modules over SRB rings,

A left Noetherian ring R is left Matlis if and only if every prime epimorphic image of R is strongly left bounded.

This thesis consists of six chapters.

In Chapter 0, we list the basic definitions and notations needed for this thesis. Some special definitions will be indicated where needed in the sequel.

In Chapter 1, we introduce basic results and examples. Some of these results are generalizations of well known results as in 1.1 and 1.11. Some examples are introduced to provide us with means of constructing SRB rings and which illustrate the distinctions between SRB rings and some other kinds of rings specifically, right duo rings as in 1.2, 1.5, and 1.7. Example 1.5, gives an interesting noncommutative ring without identity consisting of four elements which is SRB but not SLB, and also it is right duo but not left duo. It is the smallest noncommutative ring (up to isomorphism) of order four. It is shown in 1.8, that if R is SRB and $0 \neq x \in R$ is nilpotent element (that is, $x^n = 0$ for some positive integer $n > 1$), and $xR \neq 0$, then there exists a nonzero ideal M of R such that $M \leq' xR$ and $M^n = 0$. In [21, Proposition 1.3C], $R = \prod_{i=1}^{n} R_i$ is strongly right bounded if and only if each R_i, $i = 1,2,3,...,n$, is strongly right bounded. A generalization of the above result to an arbitrary direct product of rings with identity is given. In the last part of this chapter, we obtain a complete characterization of the right socle of a SRB ring which is the the direct sum of ideals of R which are minimal as right ideals and each is either nilpotent of index two or a division ring.

In Chapter 2, reduced SRB rings are investigated. Theorem 2.4, gives an important criteria for a SRB ring to be reduced: " if R is SRB, then R is reduced if and only if any finite product of right essential right ideals is a right essential right ideal ". Some connections between reduced SRB rings and various other types of rings such as subdirectly irreducibile, biregular, Baer, and polynomial identity rings are given. In particular, Theorem 2.18 is a restatement of Theorem C in [3]. Using Theorem C in [3], E.P. Armendraiz proves one of his main results [3, Theorem D] as follows: Let R be a reduced ring (with 1) satisfying a polynomial identity. If the center of R is a Baer ring then R is a Baer ring. Our theorem 2.21 generalizes the above result of E.P. Armendraiz as follows: Let R be a ring with 1. If C(R) (the center of R) is Baer and $X \cap C(R) \neq 0$ for every nonzero right ideal X of R, then R is a SRB Baer ring.

In Chapter 3, nilpotency and the SRB condition are discussed. Several preliminary results are developed which are used to prove the main result: Let R be SRB I-ring. Then the set of nilpotent elements N(R) equals the Jacobson radical J(R). In particular, if R is right Artinian, then N(R) = J(R).

In Chapter 4, we study the homomorphic image of a strongly right bounded ring. It is shown that the homomorphic image of a SRB ring is not necessarily SRB as in 4.2, 4.8, and 4.14. Some conditions under which the homomorphic image of a SRB ring is SRB are discussed as in 4.3, 4.4, 4.5, 4.6, 4.7, and 4.15. Example 4.14

illustrates that finiteness conditions have little effect on improving the SRB condition to the right duo condition or the SLB condition. It also shows that the homomorphic image of a SRB ring need not be SRB even if R is finite. It is known , in commutative rings as well as in right duo rings, that every prime ideal is completely prime but Example 4.8, shows that this is not true, in general, for SB rings. We have shown that if R is a SRB prime ring, then R is a right Ore domain as in 4.11. If R is SRB and P is a prime ideal of R which is closed as a right ideal then P is completely prime and R/P is SRB, see 4.12. One of the main results of this thesis,4.15, introduces an important tool for studying homomorphic images and prime ideals of SRB rings. It shows that if R is SRB and P is a prime ideal which is inessential as a right ideal of R, then:

(1) P is closed as a right ideal.

(2) R/P is SRB.

(3) P is a completely prime ideal of R.

This result plays an important role in the study of the decompositions of SRB right self-injective rings. It leads to one of the main results of the thesis, 4.17, which shows that if R is SRB right self-injective with identity and R has no infinite set of nonzero orthogonal idempotents, then $R = A \oplus B$, where A and B are rings such that:

(1) B is a finite direct sum of division rings.

(2) A is SRB right self-injective with identity in which every prime ideal is essential in A.

(3) $A = \oplus A_i$ (finite direct sum), where each A_i is SRB right self-injective indecomposable ring with identity in which every prime ideal is essential in A_i.

As a consequence of the above result we have the following: if R is a ring with identity, then R is a finite direct sum of division rings if and only if the following two conditions hold in R:

(1) R is SRB right self-injective.

(2) Every prime ideal is inessential.

In Chapter 5, we introduce some generalizations of R.C.Courter's results [14] concerning maximal duo algebras. He has shown that if a ring R possesses a faithful cyclic right R-module M, then R has no proper right duo overring contained in the endomorphism ring of M [14, Theorem 7]. A dual theorem concerning uniform representations is proved for rings $R \leq K_n$ [14, Theorem 8]. Theorems 5.1 and 5.10, provide information on the behavior of SRB and SLB conditions when one considers the regular representations [38, p. 82] of a ring in End(M), where M is faithful right R -module . Theorem 5.1, is a generalization of [14, Theorem 7]. Theorem 5.10, is the main result of that chapter and it is a generalization of [14, Theorem 8].

CHAPTER (0)

CHAPTER 0

BASIC DEFINITIONS AND NOTATIONS

In this chapter we list the basic definitions and notations needed for this thesis. Some special definitions will be listed where needed in the sequel.

Notations 0.1

SYMBOL	MEANING
$\mathbb{R}$	the field of real numbers.
$\mathbb{Z}$	the ring of integers.
$\mathbb{Z}_n$	the ring of integers modulo n.
$\mathbb{Q}$	the field of rational numbers.

In the sequel R will be a ring which is not necessarily commutative. R will be considered without identity unless we indicate that R is a ring with identity or with unity, (denoted by R with 1).

Definition 0.2 A nonempty subset I of R is called a right (left) ideal of R if:

(1) $(I,+)$ is a subgroup of $(R,+)$;

(2) $ar \in I$ $(ra \in I)$ for all $r \in R$ and for all $a \in I$.

If I is both a right and a left ideal of R it is called a two-sided ideal, or simply an ideal. If X is a subset of R then $\langle X \rangle_r$, $\langle X \rangle_\ell$, $\langle X \rangle$, will denote the right, left, and two-sided ideal generated by X, respectively.

Definition 0.3 A right ideal I of R is called right essential in R (denoted by I<'R) if I has a nonzero intersection with each nonzero right ideal of R. Left essential ideal is defined analogously. In the sequel" essential" will mean right essential unless indicated otherwise. I is called inessential if there exists a nonzero right ideal X of R such that $I \cap X = 0$.

Definition 0.4 A ring R is called a right (left) duo ring if every right (left) ideal of R is an ideal of R. It is called a duo ring if it is both a right and a left duo ring.

Definition 0.5 A ring R is called right (left) bounded if every essential right (left) ideal of R contains a nonzero ideal. R is called bounded if it is both right and left bounded.

Definition 0.6 A ring R is said to be fully right (left) bounded if for every prime ideal P we have R/P is right (left) bounded. R is called fully bounded if it is both fully right and left bounded.

Definition 0.7 A ring R is a strongly right (left) bounded ring, SRB (SLB), if every nonzero right (left) ideal of R contains a nonzero ideal of R. It is called strongly bounded, SB, if it is both strongly right and left bounded.

Definition 0.8 [37] The semigroup ring of a semigroup S over a ring R is the ring R[S] which consists of all formal sums of the form $\Sigma\, r_s s$ $(s \in S,\ r_s \in R)$ with evident addition and with the following rules of equality and multiplication:

$\Sigma\, r_s s = \Sigma\, r'_s s \Longleftrightarrow r_s = r'_s$ for all $s \in S$; $rs.r's = rr'.ss'$ for all $s, s' \in S$ and for all $r, r' \in R$.

Definition 0.9 If R is a ring, M is an additive abelian group and $(m,r) \longmapsto mr$ is a function from $M \times R$ into M satisfying that for all $r, r_1, r_2 \in R$ and for all $m, m_1, m_2 \in M$ we have:

i) $(m_1 + m_2)r = m_1 r + m_2 r$;

ii) $m(r_1 + r_2) = mr_1 + mr_2$;

iii) $m(r_1 r_2) = (mr_1)r_2$;

iv) $m\,1 = m$, if $1 \in R$;

then M is called a right R-module over R (denoted by M_R). If $N \subseteq M$ satisfying the following:

i) $(N,+)$ is a subgroup of $(M,+)$;

ii) $nr \in N$ for every $n \in N$ and for every $r \in R$;