كلية العلوم

ON THE HOMOLOGY AND HOMOTOPY OF FIBRE BUNDLE

THESIS

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By

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الله

بسم الله الرحمن الرحيم

قالوا سبحانك لا علم لنا إلا ما
علمتنا إنك أنت العليم الحكيم .

صدق الله العظيم
البقرة - ٣٢ -

بسم الله الرحمن الرحيم

NOTE

The present thesis is submitted to the Faculty of science, Ain Shams University in partial fulfillment for the requirements of the degree of Master of Science in Mathematics .

Besides the research work materialized in this thesis, the candidate has attended five,one-semester,post-graduate courses in the school year 1988/1989 in the following topics :

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2. Functional Analysis	2
3. General Topology	2
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5. Numerical Analysis	2

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SUMMARY

The notation of a fibre bundle first arose out of questions posed in 1930 on the topology and geometry of manifolds, [18]. By the year 1950, the definition of fibre bundle has been clearly formulated. Steenrod's book, [26], which appeared in 1950, gave a coherent treatment of the subject up to that time.

The problems connected with bundles are of various types. The problem of the great interest is that of obtaining the relations connecting the homotopy groups of the total space, base space, and the fibre. The isomorphisms between the homotopy groups of the different spaces in fibre bundle are not valid in general, although, for certain fibre bundles, the existence for such isomorphisms are possible.

The homology theory is one of the most important subjects in algebraic toplogy, its axiomatic characterization has been given firstly by S. Eilenberg and N. Steenrod in (1952), [8]. Its importance is due to the fact that an algebraic representation of topology converts toplogical problems into algebraic ones, to the end that with sufficiently many representations, the topological problems will be solvable if (and only if) all the corresponding algebraic problems are solvable. One of the most essential homology theories is the singular homology theory, [10], which is defined on any category of toplogical spaces without any restriction and always satisfies all the Eilenberg-Steenrod axioms.

The recognition of the branch of mathematics now called *homotopy theory* took place in the few years after the introduction of the homotopy groups by *Witold-Hürewicz* in (1935). Since then, with numerous advances made by various workers, it has been playing an increasing important role in the expanding field of algebraic topology, [9]. The difference between the homology and homotopy groups lies in the existence of the excision axiom in the homology theory which is replaced by the fibration axiom in the homotopy theory, [13] and [8].

There are no algorithms for computing the absolute or relative homotopy groups of topological spaces. One of the few tools available for the general study of homotopy groups is their comparsion with the corresponding integral singular homology groups. Such a comparsion is affected by means of a canonical homomorphism

$$\chi_n: \pi_n(X, A, x_o) \longrightarrow H_n(X, A)$$

from homotopy groups to homology groups which is called the *Hürewicz-homomorphism*.

For every homology structure $\mathfrak{B}$, F. W. Bauer, [4], constructed a homotopy structure $\mathfrak{B}_\pi$, in which these two structures behave to each other in the same way as the singular homology theory to the classical homotopy theory, Bauer homotopy theory is defined for a certain category of toplogical spaces and uniquely characterized through certain axioms. Moreover, a Hürewicz homomorphism $h_{\mathcal{B}}: \mathfrak{B}_\pi \longrightarrow \mathfrak{B}$ is given. The main aim of the present work is using the relation between homology and homotopy, especially the Hürewicz isomorphism, to obtain some results concerning homology from homotopy of spaces of fibre bundle, and vice versa.

The thesis consists of three chapters. The first chapter contains basic ideas from algebra, topology and algebraic topology. These ideas form the theoretical base of our study.

In the first section of chapter II, we deal with the answer of the question "*Are the homotopy groups of the total space isomorphic to those of the product spaces?*". In the second section we prove that the homotopy groups of the total space and the base space, in certain cases of fibre bundles, are isomorphic. In the third section we prove the isomorphism between the homotopy groups of the total space and those of the fibre space under certain conditions on the base space. On the other hand, the isomorphism between the homotopy groups of the base space and those of the fibre space is proved.

In the first section of chapter III, we deal with the homomorphism between homotopy group and homology group, using the classical and new ideas of Hürewicz homomorphisms. Moreover, some properties of these homomorphisms are considered. At the end of this section we prove that the classical Hürewicz isomorphism theorems are special cases of the new Hürewicz isomorphism theorems.

In the second section of chapter III, we use the classical Hürewicz isomorphism to obtain homology isomorphisms and homotopy isomorphisms for fibre bundle.

CHAPTER I

CHAPTER I

BASIC CONCEPTS

In this chapter we introduce the definitions and conventions which are required for our study. We classify them into six main sections. In the first section we consider the topological concepts, it is divided into three articles, the first article studys the basic definitions; the second article deals with the fundamental group and the third article deals with the fibre bundle and fibration. The second section is devoted to the algebraic concepts. The third section deals with the category and functors. The axioms for homology are considered in section four. We concern with the singular homology theory in section five. Finally, we consider the homotopy groups and homotopy system.

1. TOPLOGICAL CONCEPTS

This section deals with some topological constructions we classify them into three articles. In the first we consider the basic definitions and properties for spaces and maps. The second is devoted to the concept of the fundamental group, and the last deals with a brief account on the topology of fibre bundle.

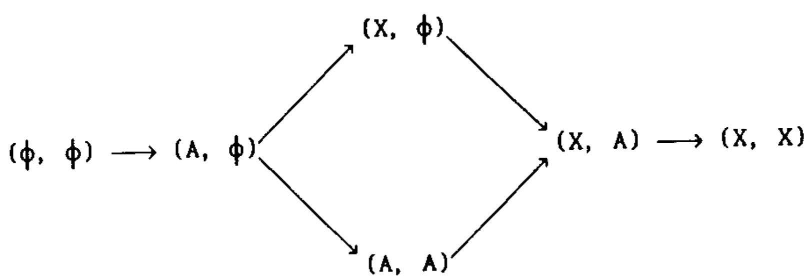
1.1 Basic Definitions:

Definition (1.1.1): A topological pair (X, A) consists of a topological space X, and a subspace A of X. If A is empty, we shall not distinguish between the pair (X, ϕ) and the space X. By a map $f: (X,A) \longrightarrow (Y,B)$ from a topological pair (X,A) into a topological pair (Y,B), we mean a (*continuous*) map $f: X \longrightarrow Y$ from the topological space X into the topological space Y satisfying $f(A) \subset B$. We shall not distinguish between the map $f: (X, \phi) \longrightarrow (Y, \phi)$ and the map $f: X \longrightarrow Y$, [17].

Definition (1.1.2): A toplogical triple(X, A, B), is a toplogical space X together with two subspaces A and B of X satisfying $B \subset A$, [17].

Definition (1.1.3): By a topological triplet (X, A, x_o), we mean a space X, non-empty subspace A of X, and a point $x_o \in A$, [13].

Definition (1.1.4): The lattice of a pair (X, A) consists of the pairs:



all their identity maps, the inclusion maps indicated by arrows, and all their compositions.

If $f: (X, A) \longrightarrow (Y, B)$, then f defines a map of every pair of the lattice of (X, A) into the corresponding pair of the lattice of (Y, B), [8].

Definition (1.1.5): Let $\{X_\alpha\}$ be a collection of sets indexed by a set M. The product $\prod_{\alpha \in M} X_\alpha$ of the collection $\{X_\alpha\}$ is the set of functions $x = \{x_\alpha\}$ defined for each $\alpha \in M$ and such that x_α, the value of x on α, is an element of X_α. In particular, if $M = \{1, 2, \ldots, n\}$, then the product $\prod_{\alpha \in M} X_\alpha$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$, $i=1, 2, \ldots, n$, and will be denoted by $X_1 \times X_2 \times \ldots \times X_n$. If each X_α is a topological space, a topology is introduced in the product of the collection $\{X_\alpha\}$ as follows:

If a finite number of x_α's are replaced by open subsets $U_\alpha \subset X_\alpha$, the product of the resulting collection is a subset of $\prod_{\alpha \in M} X_\alpha$, and is called a *rectangular open set* of $\prod_{\alpha \in M} X_\alpha$. Any union of rectangular open sets is called an *open set of the product*. The product with this topology is called the *topological product*, and the space X_α is called *α-Component of the product space*, [8].

Definition (1.1.6): Let $F = \{X_\alpha, \alpha \in M\}$ be an indexed family of spaces, and let $X = \bigcup_{\alpha \in M} X_\alpha$ be the union of all sets X_α. Define a collection τ of subsets of X as follows: A subset U of X is in τ iff $U \cap X_\alpha$ is an open set of the space X_α for every $\alpha \in M$. It is obvious that τ is a topology on X. The set X with this topology is called the *toplogical sum* or *sum of the family* F, [16].

Definition (1.1.7): Let $f: X \longrightarrow Y$ be a surjective function from a space X onto a set Y. The identification topology on Y determined by f is the topology in which $U \subseteq Y$ is closed iff $f^{-1}(U)$ is closed in X. If Y is given the identification topology, f is called an *identification map* or *projection*, [11].

Definition (1.1.8): Let R be an equivalence relation on the points of a space X. Let Y be the set of R-equivalence classes, the function k: X ⟶ Y which assigns to each point $x \in X$ its equivalence class is called the *natural projection of* X *into* Y. If Y is given the identification topology determined by k, we write Y = X/R and say that Y is the *quotient space of* X by the relation R, [11], [28] and [21].

Definition (1.1.9): Let X, Y be two given spaces and $x_o \in X$, $y_o \in Y$ be given points. Consider the topological sum $W = X \cup Y$. If we identify the point $x_o \in X$ with the point $y_o \in Y$, we get a quotient space U of W with a specified point u_o which is the class consisting of x_o and y_o. This space U will be called the *one-point union of* X and Y, and sometimes denoted by $X \vee Y$, [13]. The spaces X and Y can be considered as subspaces of $X \vee Y$ in the obvious way, and $X \vee Y$ can be considered as a subspace of the topological product $X \times Y$ by means of the imbedding

$$k : (X \vee Y, u_o) \longrightarrow (X \times Y, (x_o, y_o))$$

defined by

$$k(u) = \begin{cases} (u, y_o) & \text{if } u \in X , \\ (x_o, u) & \text{if } u \in Y . \end{cases}$$

Definition (1.1.10): The n-cube I^n is the topological product of n-copies of unit interval, i.e., $I^n = \{(t_1, t_2, \dots, t_n) \mid t_i \in I\}$. The n-cell E^n is the subset of the *euclidean n-space* R^n, defined by

$$E^n = \{x \in R^n \mid \|x\| \le 1\}.$$