THE STUDY OF SOME PROBLEMS IN GENERAL TOPOLOGY

A

THESIS

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By

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NOTE

Beside the work carried out in this thesis, the candidate has attended post graduate courses in the following topics :

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CONTENTS

I N T R O D U C T I O N

INTRODUCTION

One of the fundamental tools of research in topology and in its application to algebra is the track groups.

The track group $(P,Q)^m \equiv (P,Q)^m (X;x_o)$ is defined as the set of all homotopy classes of maps
$f: [PxI^m, \; Px\dot{I}^m \cup QxI^m] \longrightarrow [X;x_o]$, where x_o is the trivial map defined by $x_o\ (P)=x_o$, the fixed point of P.

The track groups were studied by many mathematicians such as S.T. Hu [5] in the case of P and Q are topological spaces, also Hu [5] studied the track groups when P and Q are replaced by C-W complexes.

Fox [3], had co-operated with Hu and others in simplifying the study of the relationship between various track groups and the homotopy groups.

The main object of this thesis, on one hand, is the study of the quotient track groups and obtaining some results such as the isomorphic relationship between a split extension track group and the quotient track group, beside we discuss this relationship in the status $m > 1$, and on the other hand, alternative proofs for :

(i) Hu's theorems of [5]

(ii) Fox's Lemma, are given

Also we prove some applications on Fox's Lemma.

It is necessary to consider first what is the point of difference in nature of construction between track groups and homotopy groups, the maps in track groups transforming from $\{P \times I^m, P \times \dot{I}^m \cup Q \times I^m\}$ to $\{X; x_o\}$. while in homotopy groups the maps transforming from $\{I^m, \dot{I}^m\}$ to the function compact open topological space $\{X; x_o\}^{\{P,Q\}}$.

The thesis is divided into four chapters as follows:

Chapter (I) deals with some properties about the concepts that are associated with the track groups, also the definitions of the homotopy extension theorem property and the homotopy type are given in this chapter.

Chapter (II) is devoted with the study of the exact sequence of HE-topological spaces triple $\{P,Q,R\}$, which is terminating at $(Q,R)^1$.

Chapter (III) is concerned with Fox's Lemma, and with some applications on it, where some of these applications study the relationship between track groups and the homotopy groups.

The previous three chapters are accomodated with the study of the construction of the quotient track groups and acquired that the quotient track group is a summand of a split extension track group.

Applications on the results of these chapters are proved in chapter (IV).

Some theorems on the cluster space are given in chapter (IV).

The numbers in square brackets refere to the bibliography at the end.

9

CHAPTER (I)

TRACK GROUPS

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TRACK GROUPS

In this chapter, we are firstly concerned with the concepts that are associated with track groups. Besides we study in details some related properties of the general track groups. (see [5]).

§(1.1) Basic notations

(i) Let $\{P_\lambda\}$ and $\{X_\lambda\}$ be two sets of spaces indexed by the same set Λ ; no two spaces need be disjoint.

(ii) I^m shall be the m-cube, consisting of all points $(x_1, \ldots, x_m)$ in Euclidean m-space R^m, such that $0 \leq x_i \leq 1$ for each i .

(iii) $\dot{I}^m$ is the boundary of I^m, consisting of all points $(x_1, \ldots, x_m)$ of I^m, such that $x_j = 0$ or 1 for some $1 \leq j \leq m$.

(iv) Set $E_0^{m-1} = \overline{\dot{I}^m - I^{m-1}}$, a closed set which meets I^{m-1} in $\dot{I}^{m-1}$.

(v) $\{X_\lambda\}^{\{P_\lambda\}}$ is the set of all maps of $\{P_\lambda\}$ to $\{X_\lambda\}$.

(vi) Q is a closed subspace of P.

§(1.2) Definitions

(1) A map $f : [P_\lambda] \longrightarrow [X_\lambda]$, is a set of continuous maps $f_\lambda : P_\lambda \longrightarrow X_\lambda$, such that, for all $\lambda, \mu \in \Lambda$, $f_\lambda \mid (P_\lambda \cap P_\mu) = f_\mu \mid (P_\lambda \cap P_\mu)$.

(2) If T is any topological space then a map $f: [P_\lambda] \longrightarrow [X_\lambda]$ induces a map $\hat{f}: [P_\lambda \times T] \longrightarrow [X_\lambda \times T]$ which is defined by $\hat{f}_\lambda (p,t) = (f_\lambda (p),t)$ for all $p \in P_\lambda$ and all $\lambda \in \Lambda$.

(3) The factor space P/Q (Zerlegensraum [1]) is defined by :-

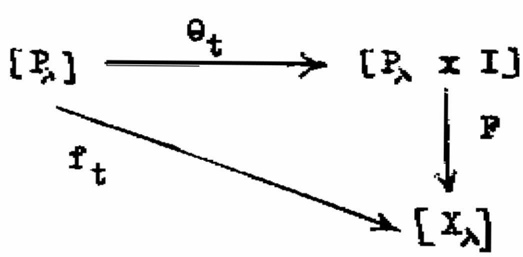
(a) If $Q = \phi$, the empty set, then $P/Q = P$, and $\emptyset : P \longrightarrow P$ is the identity map

(b) If $Q \neq \phi$, then $P/Q = (P-Q) \cup q$, where q is a point in $(P-Q)^c$, the complement set of P-Q, and $\emptyset : P \longrightarrow P/Q$ is defined by

$\emptyset \mid (P-Q) = i$, the identity map

$\emptyset \mid Q = q$

(4) If there is a map $F: [P_\lambda \times I] \longrightarrow [X_\lambda]$, such that the following diagram



is commutative, where $\theta_t : [P_\lambda] \longrightarrow [P_\lambda \times I]$ is a map defined by : $(\theta_t)_\lambda (P) = (p,t) \in P_\lambda \times I$, then we say f_o is homotopic to f_1 and we write $f_o \simeq f_1$.
The map F or the maps f_t are called the homotopy.

(5) If there is two maps $[P_\lambda] \underset{g}{\overset{f}{\rightleftarrows}} [P'_\lambda]$ such that $fg \simeq \dot{1}$ and $gf \simeq \dot{1}$, the identity maps, we say that g is a homotopy inverse of f, and that $[P_\lambda]$, $\{P'_\lambda\}$ are of the same homotopy type.

(6) The homotopy relationship between maps is an equivalence relation and divides the set $[P_\lambda]^{[X_\lambda]}$ into homotopy classes. We write $\{f\}$ for the homotopy class of f, and $(P_\lambda)^o\ (X_\lambda)$ for the set of all such classes. The addition of two classes $\{f\}$, $\{g\}$ in $(P_\lambda)^o\ (X_\lambda)$ is defined by $\{f\} + \{g\} = \{f + g\}$, where the map f+g is defined by

$$(f+g)(P,y_1,\ldots,y_m)=f(p,2y_1,y_2,\ldots,y_m) \text{ if } o \leq y_1 \leq \tfrac{1}{2}$$
$$=g(p,2y_1 - 1, y_2,\ldots,y_m) \text{ if } \tfrac{1}{2} \leq y_1 \leq 1$$

for all $p \in P_\lambda$, all $\lambda \in \Lambda$.

The homotopy class $-\{f\}$ is the inverse of $\{f\}$ contains the map $\{\bar{f}\}$ which is defined by:

$$\bar{f}(p,y_1,y_2,\ldots,y_m)=f(p,1-y_1,y_2,\ldots,y_m).$$

(7) The set $(P_\lambda)^m(X_\lambda ;x_o)$ which is <u>called the track group</u> [5] is the set of all homotopy classes of maps $f:\{P_\lambda \times \overset{m}{I}, P_\lambda \times \overset{\cdot m}{I}\} \longrightarrow [X_\lambda, x_o]$, where x_o is the trivial map defined by: $x_o(P)=x_o$ the fixed point in X_λ for all $\lambda \in \Lambda$.

(8) The m-th homotopy group $\pi_m(\{X_\lambda\}^{[P_\lambda]};\ x_o)$ is the set of all homotopy classes of maps $f\colon I^m \longrightarrow [X_\lambda]^{[P_\lambda]}$ such that $f(\dot{I}^m)=x_o$, where x_o is the trivial map from $[P_\lambda]$ to $\{X_\lambda\}$ i.e. $x_o(P_\lambda)=x_o$ a fixed point in x_λ for all $\lambda \in \Lambda$.

(9) The path component C_{y_o} of Y contains y_o , is the homotopy class contains y_o which is due to the equivalence relation, that each two points of the connected topological space Y can be connected by a path in Y.

(10) The m-th relative homotopy group ${}_m(F,G;x_o)$ is the set of all maps $f\colon\{I^m, I^{m-1}; E_o^{m-1}\} \rightarrow \{F,G;x_o\}$, where $F = \{X;x_o\}^{\{P,R\}}$, $G = \{X;x_o\}^{[P,Q]}$ and $R \subset Q \subset P$ are closed subspaces of P.

(11) A topological space X is said to be regular, if for every closed set F and every point $x \in F$, there exists an open set $G \subset F$ and a neighbourhood N_x of x , such that $G \cap N_x = \Phi$.

(12) A locally compact space, is a space which has a compact neighbourhood N_x at each point x.

(13) If $S \subset T$, and if, for any maps $f_o\colon T \longrightarrow X$, $g_o = f_o|S : S \longrightarrow x$, and for a homotopy $g_t\colon S \longrightarrow X$,