RELATIONS BETWEEN SOME CONSTANTS ASSOCIATED WITH FINITE-DIMENSIONAL VECTOR SPACES

THESIS

Submitted in Partial Fulfilment of
The Requirements for the Award
of the M. Sc. Degree

By

LABIB RASHID EL-SAYED Awad
B. Sc. Hons.

Submitted at
Faculty of Science
Ain Shams University

1983

ACKNOWLEDGEMENT

I would like to acknowledge may deepest gratitude and thankfulness to Dr. ENTISARAT MOHAMED El-SHOBAKY, associate prof. of pure Math., Ain Shams University, for suggesting the topics of the thesis, for her kind supervision and for invaluable help during the preparation of the thesis.

بسم الله الرحمن الرحيم

M.SC. COURSES

STUDIED BY AUTHOR (FEB: 1981-FEB. 1982) (AT AIN SHAMS UNIVERSITY, FACULTY OF SCIENCE):

(i) The algebraic eigenvalue problem

2 hours weekly for two semisters.

(ii) Functions of matrices

2 hours weekly for one semister.

(iii) Functional analysis I

2 hours weekly for two semisters.

(iv) Functional analysis II

2 hours weekly for one semister.

(v) Theory of integration

2 hours weekly for two semisters.

(vi) Differential Geometry

2 hours weekly for two semisters.

CONTENTS

Introduction :

There are many problems occurring in the theory of infinite-dimensional normed spaces which, appear to have only trivial counterparts in finite-dimensional Banach spaces because of the topological isomorphism of all n-dimensional normed linear spaces.

In the "local" i.e finite-dimensional, theory of Banach spaces one looks for quantitive results in places where for the infinite-dimensional analogous qualitative results are satisfactory.

Problems admitting such analogies are often rather difficult and the finite-dimensional problems do not seem much easier. But what this brings out is that we know very little about the differing metric properties of finite-dimensional spaces, and it may be that little progress will be made with some Banach space problems until more has been learned about finite-dimensional spaces.

A relevant example is that of the problem of non-equivalence of absolute and uncounditional convergence in infinite-dimensional spaces.

This was in fact demonstrated by Dvoretzky and Ragers (1950) via the estimation of a certain parameter for n-dimensional spaces.

In this thesis we concern our study to some important parameters in the finite-dimensional Lorentz sequence spaces $\ell_{p,q}^{(n)}$.

In Ch. I we introduce some notations and basic results in the Lorentz sequence spaces $\ell_{p,q}^{(n)}$.

Furthermore an important lemma on the distance of $\ell_{p,q}^{(n)}$ from $\ell_{r,s}^{(n)}$ is given.

In Ch. II we study absolutely p-summing operators in Banach spaces and we give a characterization of absolutely one summing (absolutely summing) operators in the finite dimensional Lorentz sequence spaces $\ell_{p,q}^{(n)}$.

In Ch. III and IV the main part of our thesis, we study and investigate some parameters and its relation to dimenion in Lorentz sequence spaces $\ell_{p,q}^{(n)}$. Our results improve some known results in this directions.

17

CHAPTER I

DEFINITIONS AND BASIC RESULTS

CHAPTER (I)

DEFINITIONS AND BASIC RESULTS

It is purpose of this chapter to explain certain notations and theorems used through out the present thesis.

E, F and G are always Banach spaces, with E', F' and G' we denote the topological duals of the corresponding spaces.

The set of all continuous (or bounded) linear transformations from the Banach space E into the Banach space F will be denoted by L (E, F). [29]

A linear form a on a linear space E over the field K is a mapping which determine for each element $x \in E$ a number $\langle x , a \rangle \in K$, such that

$$\langle \alpha x + \beta y , a \rangle = \alpha \langle x,a \rangle + \beta \langle y, a \rangle$$

for each $x, y \in E$ and $\alpha , \beta \in K$. [26]

For two Banach spaces E and F a mapping $T \in L(E,F)$ is called finite if its range is finite dimensional.

Each mapping of this kind can be represented in the form

$$T x = \sum_{i=1}^{n} \langle x, a_i \rangle y_i \qquad \text{for } x \in E$$

with linear forms $a_1, a_2, \ldots, a_n \in E'$ and elements $y_1, y_2, \ldots, y_n \in F$.

Jensen's inequality states that :

If $0 < p < q$, then

$$\Big(\sum_{i=1}^{\infty} |a_i|^q\Big)^{\frac{1}{q}} < \Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{\frac{1}{p}}.$$

Hoder's inequality states that :

If $1 < p < \infty$ and $q = \dfrac{p}{p-1}$,

then

$$\sum_{i=1}^{\infty} |a_i b_i| \leqslant \Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} |b_i|^q\Big)^{\frac{1}{q}}.$$

<u>Definition (1)</u> :

For $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$

the Lorentz sequence spaces $\ell_{p,q}^{(n)}$ is the set of all complex null sequences $(\xi_i)_{i=1}^{n}$ **with**

$$\|(\xi_i)_{i=1}^{n}\|_{p,q} = \Big(\sum_{i=1}^{n} i^{(q/p-1)} \, |\xi_i^*|^q\Big)^{\frac{1}{q}} \quad \text{if} \quad q < \infty$$

and

$$\|(\xi_i)_{i=1}^{n}\|_{p,q} = \max_{1 \leqslant i \leqslant n} i^{1/p} \, |\xi_i^*| \quad \text{if} \quad q = \infty$$

where $|\xi_i^*|$ denotes the i-th term in the non-increasing rearrangement of the sequence $(|\xi_i|)_{i=1}^{n}$.

Definition (2) :

A linear space X is called a quasi-normed linear space if for every $x \in X$, there is associated a real number $\|x\|$ called the quasi-norm of the vector x, which satisfies the following conditions

q_1 : $\|x\| \geqslant 0$, $\|x\| = 0 \longleftrightarrow x = 0$

q_2 : For any number λ we have

$$\|\lambda x\| = |\lambda| \|x\|$$

q_3 : For some number $\sigma \geqslant 1$, we have

$$\|x + y\| \leqslant \sigma (\|x\| + \|y\|) \quad \text{for each } x,y \in X$$

If $\sigma = 1$, the above definition given function is called a norm.

Lemma (1) : [19]

For $0 < p < \infty$, $0 < q \leqslant \infty$

$\ell_{p,q}^{(n)}$ is a quasi-normed space with respect to $\|\cdot\|_{p,q}$.

Proof :

We first prove the case when $q < \infty$

Let $(\xi_i)_{i=1}^{n} \in \ell_{p,q}^{(n)}$ with

$$\|(\xi_i)_{i=1}^{n}\|_{p,q} = \Big(\sum_{i=1}^{n} i^{(q/p-1)} |\xi_i^{*}|^{q}\Big)^{\frac{1}{q}}$$

we shall show that this function satisfies the conditions of the quasi-norm

q_1 : It is clear that $\|(\xi_i)_{i=1}^{n}\|_{p,q} \geqslant 0$

If $\|(\xi_i)_{i=1}^{n}\|_{p,q} = 0$, it follows

$$\sum_{i=1}^{n} i^{(q/p-1)} |\xi_i^{*}|^{q} = 0 .$$

consequently we have $|\xi_i^{*}| = 0$ for each $i = 1,2,\dots, n$

therefore $(\xi_i)_{i=1}^{n} = (0,0,\dots,0)$.

q_2 : For any number λ

$$\|(\lambda\xi_i)_{i=1}^{n}\|_{p,q} = \Big(\sum_{i=1}^{n} i^{(q/p-1)} |\lambda\xi_i^{*}|^{q}\Big)^{\frac{1}{q}}$$

$$= \Big(\sum_{i=1}^{n} i^{(q/p-1)} |\lambda|^{q} |\xi^{*}|^{q}\Big)^{\frac{1}{q}}$$

$$= |\lambda| \, \|(\xi_i)_{i=1}^{n}\|_{p,q}$$

q_3 : Let $(\xi_i)_{i=1}^{n}$, $(\eta_i)_{i=1}^{n} \in \ell_{p,q}^{(n)}$, therefore

$$\|(\xi_i + \eta_i)_{i=1}^{n}\|_{p,q}^{q} = \sum_{i=1}^{n} i^{(q/p-1)} |(\xi_i + \eta_i)^{*}|^{q}$$

$$\leqslant \sum_{i=1}^{n} i^{(q/p-1)} (|\xi_i^{*}| + |\eta_i^{*}|)^{q}$$

$$\leqslant \tau_q \sum_{i=1}^{n} i^{(q/p-1)} (|\xi_i^{*}|^{q} + |\eta_i^{*}|^{q})$$

where $\tau_q = \max(2^{q-1}, 1)$.

Therefore we get

$$\|(\xi_i + \eta_i)_{i=1}^{n}\|_{p,q}^{q} \leqslant \tau_q \left[\|(\xi_i)_{i=1}^{n}\|_{p,q}^{q} + \|(\eta_i)_{i=1}^{n}\|_{p,q}^{q} \right].$$

Consequently, we have

$$\|(\xi_i + \eta_i)_{i=1}^{n}\|_{p,q} \leqslant \tau_q^{\frac{1}{q}} \left[\|(\xi_i)_{i=1}^{n}\|_{p,q}^{q} + \|(\eta_i)_{i=1}^{n}\|_{p,q}^{q} \right]^{\frac{1}{q}}$$

$$\leqslant \tau_q^{\frac{1}{q}} \tau_{\frac{1}{q}} \left[\|(\xi_i)_{i=1}^{n}\|_{p,q} + \|(\eta_i)_{i=1}^{n}\|_{p,q} \right.$$

where $\tau_{\frac{1}{q}} = \max(2^{\frac{1}{q}-1}, 1)$.

Therefore

$$\|(\xi_i+\eta_i)_{i=1}^{n}\|_{p,q} \leqslant \sigma\left[\|(\xi_i)_{i=1}^{n}\|_{p,q} + \|(\eta_i)_{i=1}^{n}\|_{p,q}\right]$$

where

$$\sigma = \tau_q^{\frac{1}{q}}\,\tau_{\frac{1}{q}} = \begin{cases} 2^{\frac{1}{q}-1} & \text{for } q \leqslant 1 \\ 2^{1-\frac{1}{q}} & \text{for } q \geqslant 1. \end{cases}$$

In the second case when $q = \infty$

Let $(\xi_i)_{i=1}^{n} \in \ell_{p,q}^{(n)}$, the real valued function $\|(\xi_i)_{i=1}^{n}\|_{p,q}$

is defined as follows

$$\|(\xi_i)_{i=1}^{n}\|_{p,q} = \max_{1 \leqslant i \leqslant n} i^{\frac{1}{p}} \left|\xi_i^{*}\right|.$$

the proof of this case proceeds analogous as in the first case, therefore $\ell_{p,q}^{(n)}$ is a quasi-normed space.

Lemma (2) :

(i) For $1 \leqslant P < \infty$, $1 \leqslant q_1 \leqslant q_2 < \infty$,

then $\ell_{p,q_1}^{(n)} \subset \ell_{p,q_2}^{(n)}$, and

$$\|(\xi_i)_{i=1}^{n}\|_{p,q_2} \leqslant c\,\|(\xi_i)_{i=1}^{n}\|_{p,q_1} \text{ for each } (\xi_i)_{i=1}^{n} \in \ell_{p,q_1}^{(n)}.$$

(ii) For $1 \leqslant p_1 < p_2 < \infty$ $1 \leqslant q_1 < q_2 \leqslant \infty$

then $\ell^{(n)}_{p_1,q_1} \subset \ell^{(n)}_{p_2,q_2}$, and

$$\|(\xi_i)_{i=1}^{n}\|_{p_2,q_2} \leqslant C\,\|(\xi_i)_{i=1}^{n}\|_{p_1,q_1} ,$$

for each $(\xi_i)_{i=1}^{n} \in \ell^{(n)}_{p_1,q_1}$.

Here C is a positive constant depending on the parameters p_1, p_2, q_1, q_2 and independent of $(\xi_i)_{i=1}^{n}$.

Proof :

(i) Let $(\xi_i)_{i=1}^{n} \in \ell^{(n)}_{p,q_1}$,

$$\|(\xi_i)_{i=1}^{n}\|_{p,q_1} = \left[\sum_{i=1}^{n} i^{(q_1/p-1)} \,|\,\xi_i^{*}\,|^{q_1}\right]^{\frac{1}{q_1}} .$$

Since $q_1 < q_2$, using Jensen's inequality , we get

$$\left[\sum_{i=1}^{n} i^{(q_2/p-1)} \,|\,\xi_i^{*}\,|^{q_2}\right]^{\frac{1}{q_2}} < \left[\sum_{i=1}^{n} i^{(q_1/p-1)} \,|\,\xi_i^{*}\,|^{q_1}\right]^{\frac{1}{q_1}}$$

therefore $(\xi_i)_{i=1}^{n} \in \ell^{(n)}_{p,q_2}$ and $\ell^{(n)}_{p,q_1} \subset \ell^{(n)}_{p,q_2}$.

Since $q_1 < q_2$, we get