SOME IDEALS OF OPERATORS ON THE SPACE $\ell_{p,q}$

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M.Sc. Courses

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(i) Functional analysis I
2 hours weekly for one semister.

(ii) Functional analysis II
2 hours weekly for one semister.

(iii) Functional analysis III
2 hours weekly for one semister.

(iv) Algebraic topology
2 hours weekly for two semisters.

(v) Differential topology
2 hours weekly for one semisters.

(vi) Ordinary differential equations
2 hours weekly for one semister.

(vii) Numerical treatment of matrices
2 hours weekly for one semister.

(viii) Theory of functions of matrices
2 hours weekly for one semister.

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CONTENTS

INTRODUCTION

The theory of operator ideals, which had been initiated by the fundamental work of A. Grothendieck [7] is becoming more and more a special branch of functional analysis.

In this work we present properties of nuclear and p-nuclear operators, which are typical for the research in this branch. Then we introduce the criterions for a diagonal operator from one lorentz sequence space to another to belong to nuclear and p-nuclear ideals respectively.

1. A bounded linear operator T from a Banach space E to a Banach space F is called nuclear if there exists two sequences $\{a_i\}$ and $\{y_i\}$ of E'(the topological dual of E) and F respectively such that

$$Tx = \sum_{i=1}^{\infty} \langle x, a_i \rangle y_i$$

for every $x \in E$ with

$$\sum_{i=1}^{\infty} \|a_i\| \|y_i\| < \infty .$$

This definition is due to R. Schatten [18] and A. Grothendieck [7].

A. Tong [19] and E. El-Shobaky [3] have given a complete account of nuclear diagonal mappings from one ℓ_p space to another.

2. A bounded linear operator T from a Banach space E to a Banach space F is p-nuclear if and only if T can be factorized in the form

$$T = Q\,D\,P$$

where $P \in L(E, \ell_\infty)$ with $\|P\| \leqslant 1$, $Q \in L(\ell_p, F)$ with $\|Q\| \leqslant 1$ and D is a multiplication operator by a sequence in ℓ_p. This class of operators is introduced by A. Persson and A. Pietsch [11].

D. Garling [6] and E. El-Shobaky [3] independently have given a nearly complete account of p-absolutely summing and p-nuclear operators from one ℓ_p space to ℓ_q space with $1 \leqslant p, q < \infty$.

3. The organization of this work is the following:

In Chapter I : we collect some general results which will be used later on. Furthermore the concept of an operator ideal is introduced. Then some known classical examples are given.

In chapter II we shall present the definitions and the fundamental properties related to Lorentz sequence spaces in view of rearrangement. Especially, a result concerning the rearrangement of a sequence due to Hardy, Littlewood and Pólya, and a generalized Holders inequality will be frequently used.

In chapters III and IV we deal with nuclear and p-nuclear operators respectively. In section A of these chapters we consider the work of A. Pietsch [15] and A. Pietsch and A. Persson [11] respectively, in which they have given interesting theorems and characterization of nuclear and p-nuclear operators.

In section B of chapter III we give a new characterization of diagonal operators between Lorentz sequence spaces. In section B of chapter IV we obtain a complete account (which is new in the literature) of 2-nuclear diagonal operators between Lorentz sequence spaces.

CHAPTER I

BASIC CONCEPTS AND DEFINITIONS

In the following we introduce some concepts and definitions ([1] , [2]). E,F and G are always Banach spaces. With E', F' and G' we denote the topological duals of the corresponding spaces.

$< x, a >$ denotes the value of the bounded linear functional $a \in E'$ at $x \in E$. On E' we consider the norm

$$\|a\| = \sup_{\|x\| \leqslant 1} |< x,a >| \quad , \quad a \in E'$$

under which E' is a Banach space too.

The collection of all bounded linear operators that transform the Banach space E into the Banach space F will be denoted by L(E,F). The elements of L(E,F) are usually called operators or mappings. In the set L(E,F) two operations of addition and scalar multiplication and a norm

$$\|T\| = \sup \{ \|Tx\| , \|x\| \leqslant 1 \}$$

are defined. Under this norm L(E,F) is a Banach space.

For every operator $T \in L(E,F)$, T' (the dual operator) is an element of L(F',E') and we have

$$\|T\| = \|T'\| .$$

If the range of an operator $T \in L(E,F)$ in F is finite dimensional, then T is called a finite dimensional operator. The subset of all finite dimensional operators of $L(E,F)$ is denoted by $L_o(E,F)$. Every finite dimensional operator $T \in L_o(E,F)$ can be represented in the form

$$T x = \sum_{i=1}^{n} \langle x, a_i \rangle y_i \qquad , \quad x \in E$$

where $a_i \in E'$ and $y_i \in F$ for $i = 1,2, \ldots , n$.

For a finite dimensional operator $T \in L_o(E,F)$ which is represented by

$$Tx = \sum_{i=1}^{n} \langle x, a_i \rangle y_i$$

we associate a number given by

$$\text{trace } (T) = \sum_{i=1}^{n} \langle y_i, a_i \rangle .$$

Trace (T) is called the trace of T. Trace T is independent of the representation of T.

For $T \in L_o(E,F)$ and $S \in L(E,F)$

we have

Trace (TS) = Trace (ST).

By an operator ideal or only ideal we mean a class of bounded linear operators A, so that for all Banach spaces E and F the set

$$A(E,F) = A \cap L(E,F)$$

is a subspace of the space L(E,F). For three Banach spaces E,F and G the following ideal properties are satisfied

1) For $T \in A(E,F)$ and $S \in A(E,F)$ we have

$$S + T \in A(E,F).$$

2) For $T \in A(E,F)$ and $S \in L(F,G)$ we have

$$ST \in A(E,G).$$

3) For $T \in L(E,F)$ and $S \in A(F,G)$ we have

$$ST \in A(E,G).$$

It is clear that L(E,F) is an operator ideal and in fact it is the largest one.

The class $L_o(E,F)$ of all finite dimensional operators is also an operator ideal.

Every ideal which contains not only the null operators, contains $L_o(E,F)$ as a dense subset.

<u>Definition:</u>

A function α, which associates to every operator $T \in A$, a non-negative number $\alpha(T)$ is called an ideal norm if the following conditions are satisfied

0) If $\alpha(T) = o$ it follows that $T = o$.

N_A). For $S \in A(E,F)$ and $T \in A(E,F)$ we have

$$\alpha(S+T) \leq \alpha(S) + \alpha(T).$$

N_{I_1}) For $T \in A(E,F)$ and $S \in L(F,G)$ we have

$$\alpha(ST) \leq \|S\| \, \alpha(T).$$

N_{I_2}) For $T \in L(E,F)$ and $S \in A(F,G)$ we have

$$\alpha(ST) \leq \alpha(S) \, \|T\|.$$

An operator ideal A on which an ideal norm α is defined is called a normed ideal and is denoted by $[A, \alpha]$.

A norm ideal $[A, \alpha]$ is called complete when every component $A(E,F)$ is a Banach space.

For every one dimensional operator A we set

$$\alpha(A) = \|A\|.$$

<u>Proposition:</u>

For every operator T of the norm ideal $[A, \alpha]$ we have

$$\|T\| \leq \alpha(T).$$

<u>Proof:</u>

Since $$\|T\| = \sup_{\|x\| \leq 1} \|T x\|$$

We can take $x_\varepsilon \in E$ for every $\varepsilon > o$ such that $\|x_\varepsilon\| = 1$ and

$$\|T x_\varepsilon\| > \frac{T}{\varepsilon + 1} \qquad (1).$$

Then, there exists a one dimensional operator $S_\varepsilon \in L(E,E)$

for which

$$S_\varepsilon \; x_\varepsilon = x_\varepsilon \qquad \text{and} \qquad \|S_\varepsilon\| = 1.$$

The composite operator $T\,S_\varepsilon$ is a one dimensional operator, and from the fact that [12]

$$\alpha(T) \quad \|\tilde{T}\| = \alpha(\tilde{T}) \quad \|T\| ,$$

we have

$$\alpha\left(\frac{TS_\varepsilon}{\|TS_\varepsilon\|} \right) = C_\alpha$$

$$\alpha(TS_\varepsilon) = \|TS_\varepsilon\| \; C_\alpha \tag{2}$$

Since $TS_\varepsilon \; x_\varepsilon = T \; x_\varepsilon$ we have

$$\|T\,x_\varepsilon\| \leqslant \|T\,S_\varepsilon\| \tag{3}$$

and

$$\alpha(TS_\varepsilon) \leqslant \alpha(T) \tag{4}$$

From (1) and (3) we get

$$\|TS_\varepsilon\| > \frac{\|T\|}{1+\varepsilon} .$$

Using (3) we get

$$\alpha(TS_\varepsilon) = C_\alpha \|TS_\varepsilon\| .$$

Using (2), (4) and (1) we obtain

$$\alpha(T) \geqslant \alpha(TS_\varepsilon) = C_\alpha \|T\,S_\varepsilon\| > C_\alpha \frac{\|T\|}{1+\varepsilon} .$$

This means that

$$(1+\varepsilon) \quad \alpha(T) > C_\alpha \quad \|T\| ;$$

as $\varepsilon \longrightarrow o$ we obtain

$$\nu(T) > C_{\nu} \|T\| .$$

We can construct a new norm on the operator ideal A as

$$\nu^{*}(T) = \frac{1}{C_{\nu}} \nu(T).$$

This gives $\nu^{*}(T) \geq \|T\|$

for any operator T of the norm ideal $[A, \nu]$.

A number of examples for complete norm ideals with the operator norm as ideal norm can be found in [14].

By a quasi-norm we mean [10] a real valued function $\|\cdot\|$ defined on the space X such that the following conditions are satisfied.

1. $\|x\| \geq 0, \quad \|x\| = 0 \longleftrightarrow x = 0.$
2. For any number λ we have

$$\|\lambda x\| = |\lambda| \|x\| .$$

3. For some number $\sigma \geq 1$ we have

$$\|x + y\| \leq \sigma (\|x\| + \|y\|) \quad \forall x,y \in X.$$

If $\sigma = 1$ the above given function is called a norm.

Finally we state a useful inequality [12].

<u>Theorem:</u>

Let E be a finite dimensional space. If $T \in L(E,E)$ the following inequality is true

$$|\text{trace}\ (T)| \leq \nu(T)$$

where $\nu(T)$ is the nuclear norm of T.