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FACULTY OF ENGINEERING

BIFURCATIONS OF CONSERVATIVE RECURRENCES OF THE SECOND ORDER

BY

Eng. Makram Roshdy Eskaros

B.Sc. June 1984, Electrical Engineering Department,
Communication Section, Ain Shams University

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Supervised By

Prof. Dr.
MOHAMED IBRAHIM HASSAN

Professor of mathematics,
department of engineering
physics and mathematics,
faculty of engineering,
Ain Shams University.

Assoc. Prof. Dr.
HASSAN MOHAMED EL-HAMOULY

Associate professor of
mathematics,
Military Technical
College.

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APPROVAL SHEET

THIS THESIS

ON

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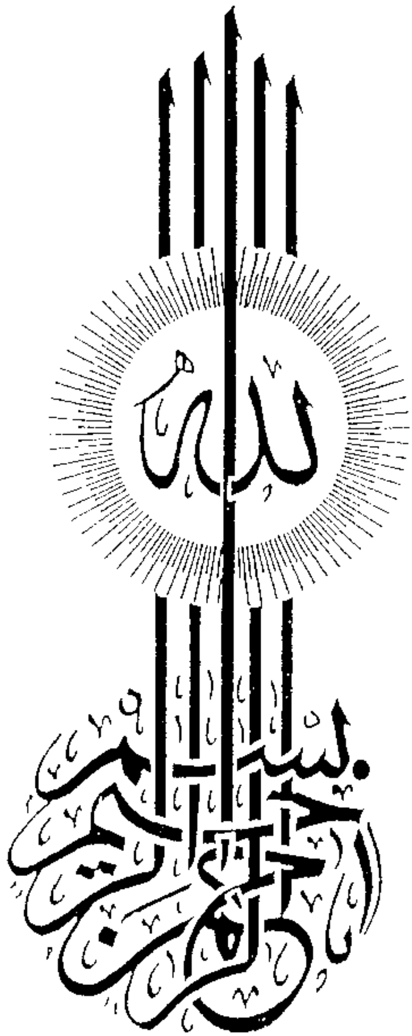
Eng. Makram Roshdy Eskaros

HAS BEEN APPROVED BY

Name, Title and Affiliation	*Signature*
1. **Prof. Dr. Edward Mekhael Ibrahim** *Professor Emeritus of Mathematics,* *Faculty of Engineering, Ain Shams University.*	E.M. Ibrahim
2. **Prof. Dr. Hasan Mustafa El-Eweidy** *Professor of Mathematics,* *Faculty of Science, El-Azhar University.*	El-Eweidy
3. **Prof. Dr. Mohamed Ibrahim Hasan** *Professor of Mathematics, Department of Engineering Physics and Mathematics* *Faculty of Engineering, Ain hams University.*	M.I. Hassan
4. **Dr. Hassan Mohamed El-Hamouly** *Associate Professor of Mathematics,* *Military Technical College.*	[signature]

Date : 2/8/1992 *Committee in charge*


بسم الله الرحمن الرحيم

STATEMENT

This dissertation is submitted to Ain Shams University for the degree of Science in Engineering Mathematics .

The work included in this thesis was carried out by the author in the Department of Engineering Physics and Mathematics, Ain Shams University from November 1987 to June 1992 .

No part of this thesis has been submitted for a degree or a qualification at any other University or institution .

Name : Makram Roshdy Eskaros

Signature :

Date : 2/8/1992

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ABSTRACT

This work aims at finding the bifurcations which happen in a second order conservative recurrence. These bifurcations take place when a parameter "a" changes in a certain domain making the behaviour of the recurrence solution undergoes a qualificative change.

Practical examples on many problems which lead to recurrence relations are introduced to show the dependence of many engineering problems on this work.

The basic theory of recurrences, its order and type, the solution and its stability, nature and type of singularities and its description in the phase plane , the possible cases of bifurcations, all have been investigated and discussed in detail by means of definitions , simple figures, various solved examples and well studied cases.

The most important part of this work is the problem of Henon which is considered here as an example of a second order conservative recurrence. Its fixed points, many cycles, and the eigenvalues and type of these singularities are obtained analytically. Also, critical and exceptional cases are discussed, many types of bifurcations are deduced and many bifurcation schemes are obtained.

The present thesis cosists of three chapters. Chapter 1 presents a general introduction to the thesis and gives some problems which lead to recurrence relations.

Chapter 2 deals with a survey of the different properties of second order autonomous recurrences of real variables.

In chapter 3 we consider the recurrence relation

$$x_{n+1}=1+y_n-ax_n^2,$$
$$y_{n+1}=-x_n,$$

which is an example of a conservative recurrence of the second order and obtain some of its singularities. Critical and exceptional cases of the recurrence under study are introduced as well as many bifurcation schemes.

Finally in appendix A we introduce the equation of transverse motion in an ideal cyclotron and show that it is possible to find a corresponding second order conservative recurrence which describes the same motion.

CONTENTS

CHAPTER 1

INTRODUCTION AND STATEMENT OF THE PROBLEM

1.1 Some problems which lead to recurrence relations

Recurrence occur in many branches of mathematics. They appear independently as natural description of time evolution phenomena or what is called dynamic systems which appear in physics, biology, electronics, ... etc.

1.1.1 Recurrence relation in connection with discrete information

An example is the command system with pulse modulation shown in fig. 1.1

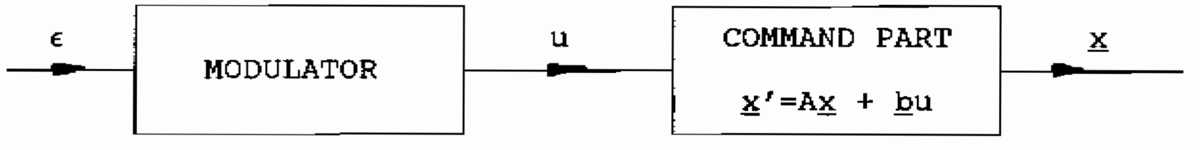

Fig 1.1

This system represents a feedback system in which the input ϵ depends continuously on the output $\underline{x}$. The error signal ϵ which is a continuous function of time is transformed into a series of pulses modulated either in amplitude or in duration or in frequency by $\epsilon(t)$. Such system is used frequently in automatic control.
The signal u is the signal of command delivered by the modulator, $\underline{b}$ is a constant vector characterizing the coupling between the modulator and the command part. The command part is supposed to be expressed by a linear differential equation

$$\underline{x}' = A\underline{x} + \underline{b}u \quad , \quad \underline{x}(t_o) = \underline{x}_o \tag{1.1}$$

where A is an (m x m) constant matrix

The solution of the linear system of equations (1.1) can be written in the form,

$$\underline{x}(t) = e^{A(t-t_o)}\underline{x}_o + \int_{t_o}^{t} e^{A(t-\tau)}\underline{b}u(\tau)\,d\tau \tag{1.2}$$

Consider the case of amplitude modulation and suppose that we have a train of pulses with period T_o and pulse duration h and amplitude function of ϵ as shown in fig. 1.2 .

We have

$$M(\epsilon) = \epsilon(t) \qquad if \;\; nT_o \le t \le nT_o + h \;\; and$$
$$= 0 \qquad if \;\; nT_o + h < t < (n+1)T_o$$

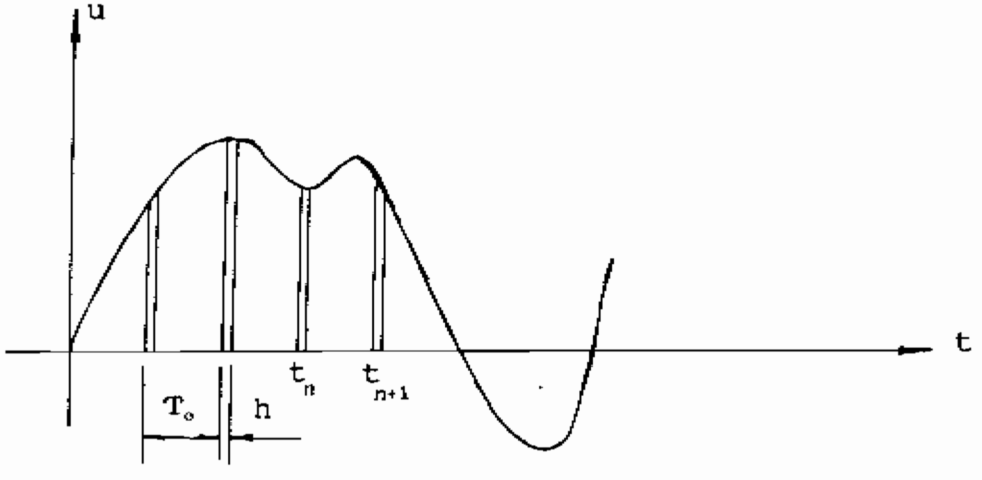

Fig. (1.2)

substituting in the equation (1.2) we get

$$\underline{x}_{n+1} = e^{AT_o}\underline{x}_n + \int_{nT_o}^{nT_o+h} e^{A[(n+1)T_o-\tau]}\underline{b}M[\epsilon(t)]\,d\tau \tag{1.3}$$

Suppose that $h<<T_o$ and the variation of ϵ is very slow then we can consider the pulses to be with constant amplitudes and we can denote $M(\epsilon)$ by M_n in the interval $nT_o < t < nT_o + h$. When A is a non-singular matrix then equation (1.3) will take the following form,

$$x_{n+1} = e^{AT_o} x_n + e^{AT_o} A^{-1} [I - e^{-Ah}] \underline{b} M_n \qquad (1.4)$$

where I is the unit matrix and A^{-1} is the inverse matrix of A.
By means of feedback in the system, M_n will be a function $\underline{x}_n$ and then we have

$$x_{n+1} = e^{AT_o} x_n + e^{AT_o} . A^{-1} . (I - e^{-Ah}) \underline{b} . f(x_n) \qquad (1.5)$$

The last equation represents a recurrence relation in m variables (the components of $\underline{x}$). It can be called an m^{th} order recurrence relation given in explicit form where T_o is a positive constant, A is a constant square matrix and $\underline{b}$ is a constant vector

1.1.2 Recurrence relation associated with differential equation

Consider the example of the electronic oscillator shown in fig.(1.3)

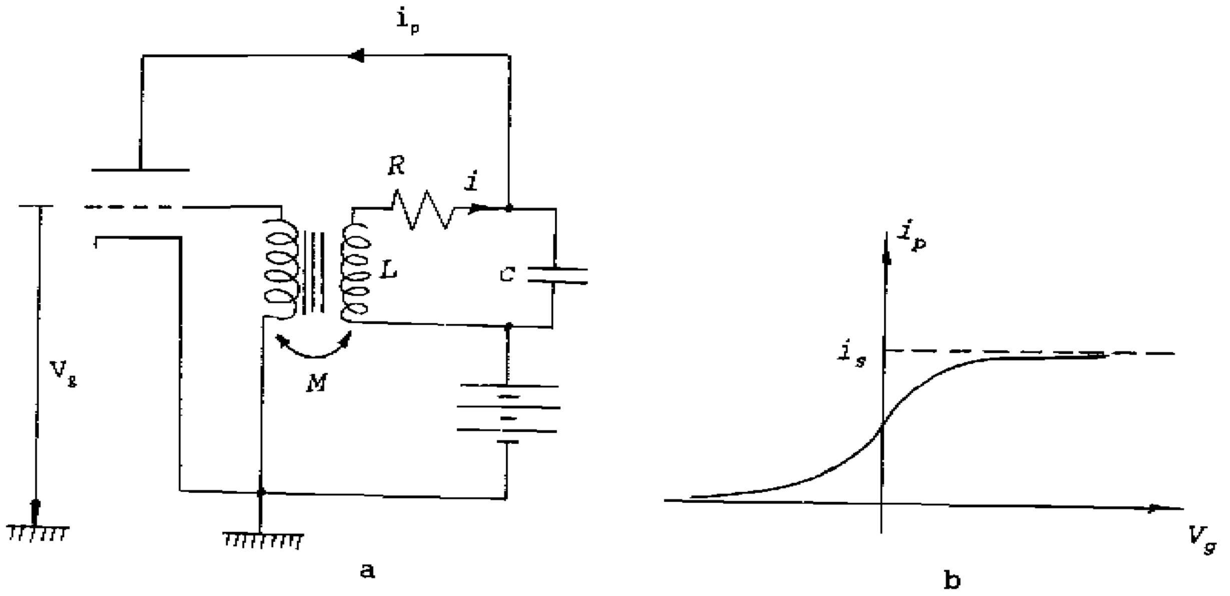

fig.(1.3)

The differential equation corresponding to this oscillator is :

$$Lci''+Rci'+i=i_p \quad , \quad V_g=-Mi' \quad , \quad i'=\frac{di}{dt} \quad , \quad i''=\frac{d^2 i}{dt^2}$$

We can write this equation as follows :

$$Lci''+Rci'-f(-Mi')+i=0 \qquad (1.6)$$

If we replace the relation $i_p = f(V_g)$ by its approximation as shown in fig. 1.3-b and if we put $\frac{i}{i_s}=x$, $\frac{1}{LC}=\omega_o^2$ and $\frac{R}{L}=2h$, we reach to the following equation

$$x'' + 2hx' + \omega_o^2 x = B \qquad (1.7)$$

Where $B = 0$ if $x' = y < 0$ and

$= \omega_o^2$ if $x' = y > 0$

It is necessary to add conditions of continuity at $y = 0$.
These conditions are due to physical considerations about the continuity of the current i and voltage v applied on the condenser c for all t including $t = 0$.
The solution of equation (1.7) can be written as follows,

$$x=Ae^{-ht}\sin[\,\omega_o\sqrt{1-\frac{h^2}{\omega_o^2}}\,t+\varphi]+a \quad , \quad a = 0 \text{ if } y < 0 \,, \quad a = 1 \text{ if } y > 0,$$

where A and φ are constants depending on initial conditions.

In the phase plane (x,y), the phase trajectory of the parametric equation $x = x(t)$ and $y = x'(t)$ are arcs of logarithmic spirals centered alternatively in $x = y = 0$ for $y < 0$ and in $x = 1$

, $\dot{y} = 0$ for $y > 0$. Let $x = x_0 > 0$ be the initial condition of equation (1.7) and $d=2\pi h(\omega_o^2-h^2)^{1/2}$ (d is the ratio of amplitudes separated by period $T=2\frac{\pi}{\omega}$ and $\omega=(\omega_o^2-h^2)^{1/2}$).

After one half oscillation in the half plane $y<0$, the trajectory initiated by $(x_0 , 0)$ cuts the x-axis in a point with abcessa $-x^*$ where $x^*=x_o\exp(-\frac{d}{2})$.

The next half oscillation $y>0$ gives a trajectory which cuts the x-axis in a point with abcessa x_1 such that $x_1-1=(x^*+1)\exp(-\frac{d}{2})$.

In fact, for $y>0$ the new origin is in $x = 1$ and the amplitudes separated by a half period are $x^* + 1$ and $x_1 - 1$ with respect to this origin. The elimination of x^* between the relations for $y<0$ and $y>0$ gives a relation between the first two successive maxima x_0 and x_1 of the curve $x = x(t)$.

This relation remains the same for every two successive maxima x_n and x_{n+1}.

$$x_{n+1}=e^{-d}x_n+1+e^{-\frac{d}{2}} \qquad (1.8)$$

This recurrence relation is linear with real variable and corresponds to the transformation of the line $y = 0$ of the phase plane into itself. The dimension of this recurrence is less by one than the dimension of the given differential equations (1.7) and permits to characterize globally the solution of (1.7). In fact, (1.8) has a fixed point $x_{n+1}=x_n=\overline{x}$ where $\overline{x}=(1-e^{-d/2})^{-1}>1$.

This fixed point corresponds to an oscillation with constant amplitude of the periodic solution $x = x(t)$ of (1.7). On the phase plane corresponding to these fixed points there exists a closed curve. This curve consists of two arcs of spirals, the first one is centered at $x = y = 0$, $y < 0$ and the other at $x = 1$, $y = 0$, $y > 0$. This curve passes through the points $x = \overline{x}$, $y = 0$ and $x=-\overline{x}e^{-d/2}$, $y = 0$.

To show that this oscillation is stable, it is sufficient to show that $\overline{x}$ is attractive. That is the sequence $\{x_n\}$ which is generated by using (1.8) by taking an initial point x_0 converges to $\overline{x}$. Let $X = x - \overline{x}$. The recurrence (1.8) will have the form

$$X_{n+1}=e^{-d}X_n \tag{1.9}$$

Since $0 < e^{-d} < 1$, it is clear to see that the sequence $\{x_n\}$ converges to $\overline{x}\ \forall\ x$.

1.1.3 Recurrence relation associated with an algebraic equation

The numerical solution of an algebraic equation or of a system of algebraic equations is made by iterative methods which associate a recurrence relation to the equation such that a stable fixed point of the recurrence relation corresponds to a solution of the algebraic equation. The most important method is Newton's method which results from the discretization with a step h of the differential equation associated with the method. Hence, applying Newton's method to the system of algebraic equations

$$p(x,y)=0 \text{ and } q(x,y)=0 \tag{1.10}$$