STORAGE-STATIONARY MODELS WITH MARKOV INPUTS

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By

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INTRODUCTION

This thesis treats some problems in the stochastic theory of storage, where we have a dam of capacity k, a random input X_t, and a release specified according to a prescribed rule. The purpose of the dam is to make the release (output) more uniform than the input in some statistical sense.

The stochastic theory of storage was first initiated by the empirical work done by Hurst (1951), but it was Moran who gave the first probabilistic model in the theory of storage (1954).

However, Moran's model was a first approximation to the real situation. He assumed that the successive inputs are mutually independent random variables, while we know in actual fact that they are correlated. He assumed that the capacity of the reservoir in infinite, while we know that it is finite.

Many authors have tried successfully to improve the assumptions of Moran's model, (Gani, Prabhu and Kendell), but all of them kept the assumption of independent inputs.

In 1964, Lloyd was able to relax the assumption of independence, replacing it by a Markovian input structure,

and thus obtaining the analogue of Morans results with independent inputs. By now Lloyd's results are known as Lloyd's model in storage theory. The development of Lloyd's model was carried by Lloyd himself, Lloyd and Odoom (1965) and Anis and Lloyd (1970, 1971).

This thesis reviews in the first two chapters Moran's and Lloyd's essential achievements for discrete inputs. The fourth chapter expounds essentially the work done by Anis and Lloyd (1971) on special Markovian input structures. But the results in the third chapter are due to the author of this thesis.

The first chapter deals with Moran's model in the case of discrete storage of infinite dams. In this chapter we study the stationary distribution of levels of water in case of independent inputs. The generating function of the stationary distribution is found for infinite dam with unit release policy. In this case the stationary distribution of storage content of a finite dam is evaluated by using the theorem of proportionality. The probability of the waiting time to first emptiness is also obtained.

The second chapter discusses Lloyd's model in the case of Markovian inputs, release unity and infinite capacity.

In this chapter we study the generating function of the stationary distribution of levels of water with serially correlated inputs (Markovian inputs). We discuss the stationary distribution of levels in a finite reservoir by using the theorem of proportionality. We also obtain the probability of emptiness in a semi-infinite reservoir with general inflow transition matrix, and the probability of waiting time to first emptiness. Finally we establish a new formulae for the mean and variance of waiting time to first emptiness in a semi-infinite reservoir with general inflow transition matrix.

In the third chapter, the investigation is carried a step further, to the case of the mean and variance of storage content (μ_z, σ_z^2) when the inputs are Markovian, the release unity and capacity infinite. These results are completely new.

Chapter four investigates special cases of storage stationary distributions with Markovian inputs. We concentrated in this chapter on a compromise model involving (as a simplification) semi-infinite capacity and (as a step towards realism) Markovian inflows. We consider for this model a number of applications of the Odoom-Lloyd matrix generating function for the asymptotic storage distribution. The results of this chapter are due to Anis and Lloyd, 1971 (3) except the last section which is due to author of this thesis.

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CHAPTER I

STATIONARY STORAGE DISTRIBUTION FOR A DAM WITH INDEPENDENT INPUTS

INTRODUCTION

The statistical treatment of water storage problems is of recent origin. The earliest work in this field is due to E.J. Gumbel (1941). He studied the return period of flood flows. Empirical work on the determination of storage capacity was done by Hurst (1951, 1956) and was extended theoretically by Anis and Lloyd (1953). In 1954 Moran gave the first probabilistic formulation of a storage model for the dam.

In this chapter we shall be concerned mainly with a mathematical discrete model on storage of water. A dam is built on a river. The reservoir is of capacity k. A random input X_t flows into the dam during the time interval $(t,t+1)$, $t = 0,1,2,\ldots$. The inputs X_t are assumed mutually independent random variables and identically distributed. A storage Z_t is defined as the dam content at time $t=0,1,2,\ldots$, the release is assumed to be instantaneous and just before reaching time t.

An overflow during the interval $(t, t+1)$ occurs if $X_t + Z_t > K$, and the amount of overflow at the end of the interval is equal to $\text{Max}(0, X_t+Z_t-K)$. The quantity of water left in the dam at the end of the interval $(t,t+1)$ and just before the release occurs, is equal to $\text{Min}(K, X_t+Z_t)$.

An amount R_n of water is to be released according to a prescribed rule. This rule is to release a quantity M of water if $X_t+Z_t \geq M$, or a quantity X_t+Z_t, if $X_t+Z_t < M$, the release being instantaneous just before the end of the time interval. From these assumptions it follows that storage function Z_t satisfies the recurrence relation

$$Z_{t+1} = \text{Min}(K, X_t+Z_t) - \text{Min}(M, X_t+Z_t), \qquad t = 0,1,\ldots$$

The purpose of the dam is to make the amount released (the output) more uniform than the input in some statistical sense. In this the following problems are of some practical importance.

1. To obtain the storage content distribution when the process described above has reached statistical equilibrium.

2. To obtain the probability distribution of the "wet period" that is the time taken for a given initial dam content to dry up. This is known as the problem of waiting period to first emptiness.

1.1- INDEPENDENT INPUTS: GENERATING FUNCTION OF LEVELS FOR "SEMI-INFINITE" DAM

The reservoirs discussed in this section are of the semi-infinite type ($K = \infty$, bounded below) introduced by Moran (9), (10), with mutually independent stationary integral-valued inputs and steady discrete outputs. The reservoir levels are observed at epochs ..., t-1, t, t+1, ... During the interval (t, t+1) a quantity X_t of water flows in, and just before the end of the interval, if there is then any water in the reservoir, a unit quantity is instantaneously withdrawn. Thus, the contents Z_t at time t are governed by the recurrence relation

$$Z_{t+1} = \text{Max}(o, Z_t+X_t-1) \quad , \quad t = o,1,2,\ldots \tag{1.1.1}$$

i.e.,

$$\begin{aligned} Z_{t+1} &= 0 \quad , \quad \text{if } X_t+Z_t \leqslant 1 \\ &= Z_t+X_t-1, \text{ if } X_t+Z_t > 1 \end{aligned}$$

It is obvious now that $\{Z_t\}$ will form an irreduciable Markov chain. Here

$$\left.\begin{aligned} p(Z_{t+1}= 0) &= p\ (0 \leqslant Z_t+X_t \leqslant 1) \ , \\ p(Z_{t+1}= r) &= p(Z_t+X_t-1= r) \ , \ r=1,2,\ldots \end{aligned}\right\} \tag{1.1.2}$$

We shall show that for the semi-infinite reservoir the sequence $\{Z_t\}$ remains ergodic provided $E(X_t) < 1$. In addition let the general distribution for the independent inputs $\{X_t\}$ be

$$p(X_t = r) = p_r \quad , r = 0,1,2,\ldots$$

and generating function

$$h(\theta) = \sum_{r=0}^{\infty} p_r \theta^r \quad , (\dot{h}(1) < 1) \tag{1.1.3}$$

The equilibrium distribution of levels $\{Z_t\}$ denoted by

$$\lim_{t \to \infty} p(Z_t = s) = u_s \quad , s = 0,1,2,\ldots \tag{1.1.4}$$

is the normalized solution of the homogeneous linear system

$$QU = U \tag{1.1.5}$$

where U is the vector $\{u_s;\ s=0,1,2,\ldots\}$, and Q is the transition matrix $Q = (q_{rs})$ where

$$q_{11} = p_0 + p_1 \quad , q_{rs} = p_{r-s+1} \text{ for } r > 1,\ s \leq r+1 \tag{1.1.6}$$

$$q_{rs} = 0 \quad \text{otherwise}$$

Hence from equation (1.1.5) and (1.1.6) we have

$$\left.\begin{aligned} u_0 &= (u_0 + u_1)p_0 + u_0 p_1 \\ u_r &= \sum_{s=0}^{r+1} u_s p_{r+1-s} \quad , r = 1,2,\ldots \end{aligned}\right\} \tag{1.1.7}$$

If the generating function of the asymptotic equilibrium distribution of levels u_r, $r=0,1,2,\ldots$, is

$$g(\theta) = \sum_s u_s \theta^s$$

then, for a semi-infinite reservoir (bounded below) one may multiply as given in (1.1.7) by θ^s and sum over all possible values of s to obtain,

$$g(\theta)\{ h(\theta)- \theta \}= (1- \theta)u_o p_o ,$$

whence, making use of the fact $g(1) = 1$, we find

$$u_o p_o = 1 - \mu_x \qquad (1.1.8)$$

where $\mu_x = E(X_t) < 1$. Thus the generating function of u_s is

$$g_z(\theta) = (1- \mu_x)(1 - \theta)/\{ h(\theta)- \theta\} \qquad (1.1.9)$$

We thus have a simple expression for the asymptotic "probability of emptiness of the semi-infinite reservoir", viz

$$u_o = (1- \mu_x)/p_o , \quad (\mu_x < 1) \qquad (1.1.10)$$

The mean and variance of Z_t may be also obtained from (1.1.9) by simple differentiation at $\theta = 1$.

This gives us

$$g_z^{\prime}(1) = (\sigma_x^2 + \mu_x^2-\mu_x)/2(1- \mu_x) \qquad (1.1.11)$$

$$g_z^{\bullet}(1)=\mu_z(\sigma_x^2+\mu_x^2-\mu_x)/(1-\mu_x)+h^{(3)}(1)/3(1-\mu_x) \qquad (1.1.12)$$

where $h^{(3)}(1)=\sum_{r=o}^{\infty} r(r-1)(r-2)p_r$,

$$=\mu_x^{(3)}-3(\sigma_x^2+\mu_x^2-\mu_x)-\mu_x$$

where

$$\mu_x^{(3)}=\sum_{r=o}^{\infty} r^3 p_r$$

and hence $\mu_z = E\{Z_t\} = g_z^{\bullet}(1)$

$$\sigma_z^2= g_z^{\bullet\bullet}(1)+ g_z^{\bullet}(1) - g_z^{\bullet 2}(1)$$

$$= \mu_z(\mu_z+1)+(\mu_x^{(3)}-\mu_x)/3(1-\mu_x) \qquad (1.1.13)$$

1.2- THE THEOREM OF PROPORTIONALITY

In a finite reservoir, of capacity K(K is a positive integer), and therefore of maximum level K-1, the distribution of level, say

$$u_s^{(k)}= \lim_{t\to\infty} p(Z_t = s|Z_{max}= K-1), \qquad (1.2.1)$$

is obtained by solving the equations

$$Q^{(k)}\ U^{(k)} = U^{(k)} \qquad (1.2.2)$$

where $U^{(k)}$ is the vector $\{u_s^{(k)};\ s = o,1,2,\dots, k\}$, and the matrix $Q^{(k)}$ consists of the first rows K-1 and first K columns