ON THE INVARIANT SUBSPACE PROBLEM

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By

EL - SAYED MOHAMED HENDY

M. Sc
15729

SUBMITTED AT

AIN SHAMS UNIVERSITY
FACULTY OF SCIENCE

Head of the Dept.
prof. M. S. Marly

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بسم الله الرحمن الرحيم

M.Sc. COURSES

STUDIED BY THE AUTHOR (1980 - 1981)
AT AIN SHAMS UNIVERSITY
FACULTY OF SCIENCE.

1) Algebraic Topology.	2 hours weekly
2) Differential Geometry.	2 hours weekly
3) Functional Analysis.	2 hours weekly
4) Numerical Analysis.	2 hours weekly
5) Spectral Theory.	2 hours weekly

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CONTENTS

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INTRODUCTION

In this thesis we present a survey of status of the invariant subspace problem. The invariant subspace problem is the question whether every operator in $\mathcal{L}(H)$ has a nontrivial invariant subspace, where H is a complex Hilbert Space of infinite dimension.

In chapter (I) we shall discuss the status of the invariant subspace problem in case of normal operators, and from the spectral theorem, which exhibits normal operators as integrals with respect to some of their reducing subspaces, we get the existence of the hyperinvariant subspaces for the normal operators (theorem 12).

In chapter (II) we shall study two different kinds of operators: Bilateral shifts, which are unitary operators, and unilateral shifts, which are restrictions of bilateral shift to certain invariant subspaces. Then we shall describe the invariant and reducing subspaces for them.

In chapter (III) we are concerned with a quasitriangular operators. At first we give a summary of some results known for compact operators, then we study the class of all quasitriangular operators and invariant subspaces associated with them in some special cases.

In Chapter (IV) we single out a special sub-class of the invariant subspaces which is called the inner invariant subspaces and this chapter can be divided into three sections:

In section (1) we present the definitions of the invariant and inner invariant subspaces for the operators (not necessarily bounded) and then we study their fundamental properties.

In section (2) we calculate the inner invariant subspaces of the shift operator in several settings. In one setting we describe the inner invariant sub-spaces of the shift on the Hardy spaces H^p, and then we generalize this result. In the last section we present some applications.

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CHAPTER 0

SOME BASIC DEFINITIONS AND RESULTS

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(1) Symboles:

We begin by some symboles used in this thesis

H. Hilbert space.

$(.,.)$ inner product.

$H \oplus K$ direct sum of H and K.

$\vee \mathcal{L}$ the span of $\mathcal{L} \subset H$.

$M^{\perp}$ the orthogonal complement of $M \subset H$.

$\mathcal{L}(H,H)$ the space of all linear operators on H.

$B(H)$ the Banach algebra with identity 1.

P is a projection operator.

U Unitary operator.

Lat A the set of all subspaces of H invariant under A.

$\sigma(x)$ the spectrum of x ; $\sigma(x) = \{\lambda \in C , (x-\lambda)$ has no inverse$\}$.

$\rho(x)$ the resolvent set ; $\rho(x) = (\sigma(x))'$

$r(x)$ the spectral radius,

$$r(x) = \sup \{ |\lambda|, \lambda \in \sigma(x)\}$$

$\pi_0(A)$ the point spectrum of A,

$$\pi_0(A) = \{\lambda \in C ; (A-\lambda) \text{ is not one to one}\}$$

$\lambda \in \pi_0(A)$ implies $\exists x$ such that $Ax = \lambda x$.

$\pi(A)$ the approximate point spectrum of A,

$$\pi(A) = \{\lambda \in C , (A-\lambda) \text{ is not bounded below}\}$$

$\Gamma(A)$ the compression spectrum of A,

$$\Gamma(A) = \{\lambda \in C \text{ such that the range of } (A-\lambda) \text{ is not dense}\}$$

$\mu(A)$ the measure of a set A.

(X,S,μ) the measurable space.

$\bar{R}$ or $\{R\}^-$ the closure of R in Hilbert space H.

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(2) Some basic definitions and results:

We shall be concerned with operators on the unique (up to isomorphism) separable, infinite dimensional complex Hilbert space H.

1. A unitary operator U is an automorphism of H i.e. $U \in B(H)$ is unitary iff U is invertable and $(Ux, Uy) = (x, y)$, $x, y \in H$.

2. A is unitarly equivalent to B if there is U such that $A = UBU^{-1}$.

3. The subspace $M \subset H$ is called invariant subspace under the operator A if $Ax \in M$ whenever $x \in M$.

4. The collection of all subspaces of H invariant under A is denoted by Lat A.

5. The vector $x \in H$ is cyclic for the operator A if $V\{A^n x\} = H$, and the subspace $M \subset H$ is called a cyclic subspace for A if $V\{A^n x\} = M$.

6. The subspace M is said to be hyperinvariant subspace for A if $M \in$ Lat B for every B satisfies $AB = BA$.

7. The subspace $M \subset H$ is reduces A if M and $M^{\perp}$ are both in Lat A.

(3) Some basic theorems:

Theorem ([8] or [16]) (1):

If $A \in B(H)$ and P is any projection onto M, then $M \in$ Lat A iff $AP = PAP$.

Theorem [16] (2):

If $A \in B(H)$ and P is the projection on M along N then M and N are both in Lat A iff $AP = PA$.

Remark:

1. It is easy to see that M reduces A iff $M \in$ (Lat $A \cap$ Lat A^*) where Lat $A^* = \{M : M^{\perp} \in \text{Lat } A\}$
2. M reduces A iff the projection P on M commutes with A.

Theorem [18] (3):

If $\mathfrak{U}$ is a complex, commutative Banach algebra with identity and $x \in \mathfrak{U}$, then

$$\sigma(x) = \{\phi(x) : \phi \text{ is a homomorphism of } \mathfrak{U} \text{ onto } C\}.$$

Theorem (4):

If B is a maximal commutative subalgebra of the complex Banach algebra $\mathfrak{U}$, if $x \in B$, then $\sigma_{\mathfrak{U}}(x) = \sigma_{B}(x)$ where $\sigma_{\mathfrak{U}}(x)$ and $\sigma_{B}(x)$ denote the spectra of x relative to $\mathfrak{U}$ and B.

Proof:

Let $x \notin \sigma_{B}(x)$, then $x-\lambda$ has an inverse in B, but B is subalgebra of $\mathfrak{U}$, then $x-\lambda$ has an inverse in $\mathfrak{U}$ implies that $x \notin \sigma_{\mathfrak{U}}(x)$. Therefore we conclude that $\sigma_{\mathfrak{U}}(x) \subset \sigma_{B}(x)$.

Suppose that $(x-\lambda)y = y(x-\lambda) = 1$ for some $y \in \mathfrak{U}$.

Then, for each $z \in B$, $yz(x-\lambda)y = y(x-\lambda)zy$, so that $yz = zy$. Hence y commutes with every member of B, and by maximality of B, $y \in B$. Thus $\sigma_{B}(x) \subset \sigma_{\mathfrak{U}}(x)$.

Theorem (5) (Spectral Mapping Theorem):

If $x \in \mathfrak{U}$ and p is any polynomial, then

$$\sigma(p(x)) = \{ p(\lambda) \ : \ \lambda \in \sigma(x)\} .$$

Proof:

Let B be any maximal commutative subalgebra of $\mathfrak{U}$ containing x. Then, by Theorems (3), (4)

$$\sigma(p(x)) = \sigma_{B}(p(x)) = \{\phi(p(x)) : \phi \text{ a homomorphism of } B \text{ onto } C\} .$$

$= \{p(\varphi(x)) : \varphi$ a homomorphism of $\mathcal{B}$ onto $C\}$.

$= \{p(\lambda) : \lambda \in \sigma(x)\}$.

It is easy prove that $\pi_0(A) \subset \pi(A)$ and $\pi(A) \cup \Gamma(A) \subset \sigma(A)$.

Theorem (6):

If $A \in B(H)$, then $\pi(A) \cup \Gamma(A) = \sigma(A)$

Proof:

We need only show that $\sigma(A) \subset \pi(A) \cup \Gamma(A)$ i.e. if $\lambda \notin \pi(A)$ and $\lambda \notin \Gamma(A)$ implies $\lambda \notin \sigma(A)$. But $\lambda \notin \pi(A)$ implies as is easily verfied by considering sequential limit points, that the range of $(A-\lambda)$ is closed. Since $\lambda \notin \Gamma(A)$, then the range of $(A-\lambda)$ is also dense , and it follows that $(A-\lambda)$ is (1-1) and onto, and thus invertible i.e. $\lambda \notin \sigma(A)$.

The next result is useful for the study of invariant subspaces. For subsets S of C we use ∂S to denote the point-set boundary of S; i.e., $\partial S = \bar{S} \cap (\overline{C \setminus S})$.

<u>Theorem (7)</u>:

If $A \in \mathcal{B}(H)$, then $\partial\sigma(A) \subset \pi(A)$

<u>Proof</u>:

First, from definition, $\partial\sigma(A) = \overline{\sigma(A)} \cap \overline{(C \setminus \sigma(A))}$. Since $\sigma(A)$ is closed set, then $\sigma(A) = \overline{\sigma(A)}$, and $\overline{(C \setminus \sigma(A))} = \overline{\rho(A)}$ Then we have $\partial\sigma(A) = \sigma(A) \cap \overline{\rho(A)}$.

We want to show that $\partial\sigma(A) = \sigma(A) \cap \overline{\rho(A)} \subset \pi(A)$. Let $\lambda \in \partial\sigma(A)$, and we want to show that $\lambda \in \pi(A)$.

Let the contrary i.e. let $\lambda \notin \pi(A)$. Choose a sequence $\{\lambda_n\} \subset \rho(A)$ which converges to λ. We claim that there is a $k > 0$ and a positive integer N such that $n \geqslant N$ implies $\|(A-\lambda_n)\, x\| \geqslant k\, \|x\|\ \forall x$. If this case were not, then for all positive integers m and N there would exist an $n \geqslant N$ and a vector x_m of norm 1 such that $\|(A-\lambda_n)x_m\| < \frac{1}{m}$. But

$$\|(A-\lambda)x_m\| \leqslant \|(A-\lambda_n)x_m\| + \|(\lambda-\lambda_n)x_m\|$$
$$\leqslant \frac{1}{m} + |\lambda - \lambda_n| \quad ,$$

and this implies $\lambda \in \pi(A)$. Hence there do exist such k and N. We can now show that $\lambda \notin \Gamma(A)$. For if $x \in H$, then for each n there is a $y_n \in H$ with $(A-\lambda_n)y_n = x$.

Since $\|(A-\lambda_n)y_n\| \geqslant k\|y_n\|$, this gives $\|x\| \geqslant k\,\|y_n\|$ i.e.
$\|y_n\| \leqslant (1/k)\,\|x\|$ for $n \geqslant N$. Then

$$\begin{aligned}\|(A-\lambda)y_n - x\| &\leqslant \|(A-\lambda_n)y_n - x\| + \|(\lambda_n - \lambda)y_n\| \\ &= 0 + |\lambda_n - \lambda|\,\|y_n\| \\ &\leqslant \frac{1}{k}\,\|x\|\,|\lambda_n - \lambda|\end{aligned}$$

If n is sufficiently large, $\frac{1}{k}\,\|x\|\,|\lambda_n - \lambda|$ is arbitrary small, then the range of $(A - \lambda)$ is dense in H, implies that $\lambda \notin \Gamma(A)$. Then we have $\lambda \notin \pi(A) \cup \Gamma(A) = \sigma(A)$ (Theorem 6) which contradiction with $\lambda \in \partial\sigma(A)$.

Difinition (1):

If $A \in B(H)$, then the full spectrum of the operator A, denoted by $\eta(\sigma(A))$, is the union of $\sigma(A)$ and all the bounded components of $\rho(A)$; i.e., $\eta(\sigma(A))$ is $\sigma(A)$ together with the "holes" in $\sigma(A)$.

This theorem gives information about the spectra of restrictions to invariant subspaces.

Theorem ([5] or [19])(8):

If $M \in$ Lat A, then $\sigma(A|M) \subset \eta(\sigma(A))$

Proof:

First note that $\lambda \in \pi(A|M)$ implies that $(A-\lambda)$ is