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ON SYMMETRIC STRUCTURES IN BANACH SPACES

THESIS

Submitted in Partial Fulfilment of
The Requirements for the Award
of the M. Sc. Degree

By

IBRAHIM HOSSNEY IBRAHIM

B. Sc.

Submitted to
Faculty of Science
Ain Shams University

1984

الله

بسم الله الرحمن الرحيم

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"رب اوزعنى أن اشكر نعمتك التى أنعمت على " وعلى والـــدى ...
وأن اعمل صالحا ترضاه .... وادخلنى برحمتك فى عبادك الصالحين" .

"صدق اللــه العظيـــم"

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* سورة النمـــل ١٩ .

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M.SC. COURSES

STUDIED BY AUTHOR (FEB. 1981 - FEB. 1982) (AT AIN SHAMS UNIVERSITY, FACULTY OF SCIENCE):

(i) The algebraic eigen value problem

2 hours weakly for two semisters.

(ii) Functions of matrices

2 hours weakly for one semister.

(iii) Functional analysis I

2 hours weakly for two semisters.

(iv) Functional analysis II

2 hours weakly for one semister.

(v) Theory of integration

2 hours weakly for two semisters.

(vi) Differential Geometry

2 hours weakly for two semisters.

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أستاذ مساعد بقسم [illegible]
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CONTENTS

The present thesis attempts to use the methods and knowledge accumulated in this field to investigate a class of spaces with a symmetric structure namely the Orlicz sequence space ℓ_M.

In chapter I we summarize some known definitions and results on schauder basis and symmetric basis.

In chapter II we are dealing namely with the convex functions and its role in the investigation of the structure of the sequence space (13, 14, 15, 16, 17, 18, 19 and 20) and give some examples of Orlicz spaces.

In chapter III we study one of the most important properties of basis in Banach spaces, namely stability of basis in Banach spaces and we obtain necessary and sufficient conditions on a unit vector basis of ℓ_M to be stable.

CHAPTER (I)

Basis in Sequence Spaces

§1. Basic Definitions and Results

Definition 1.

Let X be a Banach space and (x_n) be a sequence in X. We say that (x_n) is a Schauder basis for X if for each $x \in X$ there is a unique sequence (a_n) of scalars such that $x = \sum_{n=1}^{\infty} a_n x_n$. (where the convergence is taken in norm).

Clearly if X has a Schauder basis, then X is separable, since the set of all finite linear combinations $\sum_{i=1}^{n} r_i x_i$, where r_i are rational numbers and n = 1,2,... is a countable dense set in X. The converse has been a celebrated question for many years. In (1973) Per Enflo [7] has shown that there is indeed a separable Banach space without a Schauder basis (in fact, his example shows there is a Banach space which fails to possess a weaker property "the Grothendieck approximation property").

Theorem 1.

Let X be a Banach space and suppose that (x_n) is a Schauder basis for X. Then for each n, the functional x_n^* defined by

$$x_n^* \left(\sum_{k=1}^{\infty} a_k x_k \right) = a_n$$

is continuous.

Proof :

Let $P_n : X \longrightarrow X$ be defined by

$$P_n \left(\sum_{k=1}^{\infty} a_k x_k \right) = \sum_{k=1}^{n} a_k x_k \quad .$$

Clearly P_n is a linear projection on X and for each $x \in X$,

$$\rho(x) = \sup_n \| P_n(x) \| < \infty \quad .$$

Moreover, ρ is a norm on X and clearly $\rho(x) \geqslant \|x\|$ for all $x \in X$. We shwo that ρ is an equivalent norm on X. It is sufficient to show that ρ is a complete norm. Let (y_m) be a Cauchy sequence with respect to ρ. Then for $j > k$,

$$\|(P_j - P_k)(y_m - y_n)\| \leqslant \| P_j(y_m - y_n) \| + \| P_k(y_m - y_n) \|$$
$$\leqslant 2 \; (y_m - y_n).$$

In particular, for $j = k + 1$ we get that

$$\| x_j^* \,(y_m - y_n)\; x_j \| = | x_j^* \,(y_m - y_n) | \; \| x_j \|$$
$$\leqslant 2 \; (y_m - y_n).$$

Thus

$$\lim_{m \to \infty} x_j^* \,(y_m)\; x_j = a_j x_j \quad \text{for some } a_j \, .$$

Now, let $\varepsilon > 0$ be given and choose m_0 such that if $n > m \geqslant m_0$, then $\rho(y_m - y_n) < \varepsilon$. Then by taking the limit on n we see that

$$\left\| (P_j - P_k)\, y_m - \sum_{i=k+1}^{j} a_i x_i \right\| \leq 2\,\varepsilon .$$

Moreover,

$$\lim_{k \to \infty} \left\| P_k\,(y_m) - y_m \right\| = 0$$

implies that there is a k_0 such that if $j > k \geq k_0$, then

$$\left\| (P_j - P_k)\, y_m \right\| < \varepsilon \qquad (m \geq m_0).$$

Thus we get that

$$\left\| \sum_{i=k+1}^{j} a_i x_i \right\| < 2\,\varepsilon \qquad \text{when} \quad j > k \geq k_0 .$$

By the completeness of the original norm on X, $\sum_{i=1}^{\infty} a_i x_i$ converges to some element $y \in X$. By the uniqueness of the expansion of y, $a_i = x_i^*(y)$ and

$$(P_j - P_k)\, y = \sum_{i=k+1}^{j} a_i x_i .$$

Thus $\rho\,(y_m - y) \leq \varepsilon$ when $m \geq m_0$ and it follows that ρ is complete.

To see that each x_n^* is continuous note that

$$\left| x_n^*\,(x) \right| = \left\| P_n(x) - P_{n-1}(x) \right\|$$
$$\leq 2\,\rho\,(x) \qquad \text{when } \|x\| = 1.$$

We say that the sequence (x_n^*) is biorthogonal to the sequence (x_n).

Definition 2.

Let X be a Banach space. A sequence (x_n) in X is said to be a basic sequence if (x_n) is a Schauder basis for the closed linear span $[x_n]$ of (x_n).

The next theorem is useful for determining when sequences are basic.

Theorem 2.

Let (x_n) be a sequence of non-zero vectors in a Banach space X. Then (x_n) is a basic sequence if and only if there is a $K \geq 1$ such that for each finite sequence $a_1, a_2, \ldots, a_n$ of scalars,

$$\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i x_i \right\| \qquad \text{for all } m \leq n.$$

Proof:

Suppose (x_n) is a basic sequence. Then the norm defined in the previous theorm is an equivalent norm to the original norm on $[x_n]$.

In particular, there is a $K \geq 1$ such that for each sequence (a_n) such that $\sum_{n=1}^{\infty} a_n x_n$ exists in X,

$$\sup_n \left\| \sum_{j=1}^{n} a_j x_j \right\| \leq K \left\| \sum_{j=1}^{\infty} a_j x_j \right\| .$$

Conversely, if the condition holds, then by induction it is easy to see that expansions in terms of the x_n's are unique.

Suppose $x \in [x_n]$ i.e.

$$x = \lim_n \sum_{i=1}^{m_n} a_{i,n} \; x_i \qquad \text{for some } a_{i,n}.$$

Then for $n > m$,

$$\left\| \sum_{i=1}^{m} (a_{i,n} - a_{i,m}) \; x_i \right\| \leq K \left\| \sum_{i=1}^{n} a_{i,n} \, x_i - \sum_{i=1}^{m} a_{i,m} \, x_i \right\| .$$

This means that $(a_{i,n})$ is a Cauchy sequence for each i. Then following the proof of theorem (1), we have

$$x = \sum_{n=1}^{\infty} a_n x_n ,$$

where $a_n = \lim_i a_{i,n}$ for all n.

Let X and Y be Banach spaces and suppose that (x_n) and (y_n) are basic sequences in X and Y respectively. The sequences (x_n) and (y_n) are said to be equivalent provided that a series $\sum a_n x_n$ converges in X if and only if the series $\sum a_n y_n$ converges in Y. We shall write $(x_n) \approx (y_n)$. The two series are called fully equivalent if and only if there is a linear isomorphism $T : [x_n] \longrightarrow [y_n]$ defined by

$$T\left(\sum a_n x_n\right) = \sum a_n y_n.$$

We shall write $(x_n) \approx (y_n)$ in this case.

Definition 3.

Let (x_n) be a basic sequence in X, (p_n) a strictly increasing sequence of positive integers, and (a_n) a sequence of scalars. The sequence (y_n) defined by

$$y_n = \sum_{i=P_n+1}^{P_{n+1}} a_i x_i$$

is called a block basis with respect to (x_n).

Theorem 3. [3]

Let X be a Banach space, (x_n) a Schauder basis for X, and (x_n^*) the biorthogonal sequence to (x_n). If (y_n) is a sequence satisfying $\inf \| y_n \| = \varepsilon > 0$ and

$$\lim_n x_i^* (y_n) = 0 \qquad \text{for all } i,$$

then there is a subsequence of (y_n), which is equivalent to a block basis with respect to (x_n).

Proof :

Let K be the constant given in theorem (2) for (x_n). Choose strictly increasing sequences (P_n) and (q_n) of positive