ON IDEALS OF OPERATORS WITH TRACE

THESIS

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BY

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To

My

Parents

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M. Sc. COURSES

	hours/week
1. Functional analysis.	2
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INTRODUCTION

In this thesis, we are mainly interested in studying the trace of operators in Hilbert spaces and Banach spaces. In fact, the concept of trace plays an important role in several mathematical branches; especially in partial differential equations. In our study, we discuss the definition of trace in its different forms and study its main properties in different spaces. Our study is mainly devoted to trace of matrices, hence finite dimensional operators, and finally to nuclear operators in Banach spaces. In this direction, we have discussed what type of operators can possess a trace and consequently discussed the question of summability of the sequence of eigenvalues of an operator. This motivated us to follow the relation between eigenvalues of an operator and its classical s-numbers. In this concern, we have tried to turn the light on the way that the definition of classical s-numbers is generalized to give the definition of s-numbers of operators in Banach spaces.

Following this study one can reach to the conclusion that, in Hilbert spaces, although one can define a trace for operators which are not nuclear (see section 3.2) however, the class of nuclear operators is a very suitable class to define a trace on. In our opinion, the above conclusion is very natural since the definition of nuclear operators is formally a very natural extension of finite dimensional operators which of course possess a trace.

The question is different in Banach spaces; where generalizing the notion of a trace is not guaranteed even for nuclear operators.

In Chapter IV, we consider the work of A. Pietsch [15] in which he gives an appropriate generalization of trace on operator ideals. We also discuss some questions related to radicality of operator ideals, Gohberg operators, and operators of Riesz type ℓ_p .

Parallel to s-numbers of operators we study in Chapter V the entropy numbers of operators in Banach spaces. In this direction we have obtained a lower estimation of outer and inner entropy numbers of a matrix operator in case of finite dimensional spaces ℓ_1^n , ℓ_∞^n , (see [21]). The obtained estimation generalizes a known result for multiplication operators [14]. We introduce a sequence of matrix operators in a way similar to that of Littlewood which gives a slight modification having similar properties. Our introduced sequence of matrix operators provides an additional property; namely its elements commute with certain operators of multiplication.

A reference number such as 2.5.4 means item number 4 in section 5 in Chapter II.

Where it has seemed helpful to mask the end of a proof we have used the symbol "#".

CHAPTER I

TRACE OF FINITE DIMENSIONAL OPERATORS

Introduction :

In this chapter, we study and verify the equivalence of different definitions of a trace of a square matrix. We also study properties of the traces of matrices such as trace of summation and multiplication of operators and so on. We study a type of multiplication of matrices called Kronecker product and we give some examples of matrices that their traces satisfy certain additional properties. At the end of this chapter we study the trace of finite dimensional operators and give an example of an operator to illustrate the fact that its trace is the same as that of its corresponding matrix.

1.1 TRACE OF MATRICES

1.1.1. Definition [5]

Let $A = [a_{ij}]$ be a square matrix of order n . Then the sum of the eigenvalues λ_i of the matrix A is called its spectral trace, written

$$\lambda(A) = \sum_{i=1}^{n} \lambda_i \qquad .$$

This number is the minus of the coefficient of λ^{n-1} in the expansion :

$$\det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$$

$$= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n \qquad (1)$$

i.e. $\lambda(A) = -c_1$.

1.1.2. Remark

Setting $\lambda = 0$ in (1) gives

$$\det(-A) = (-\lambda_1) \ldots (-\lambda_n).$$

But

$$\det(-A) = (-)^n \det A .$$

Therefore ,

$$\det A = \lambda_1 \lambda_2 .. \lambda_n \quad . \tag{2}$$

1.1.3. Proposition [5]

Let $A=[a_{ij}]$ be a square matrix of order n. Then the following, are equal forms of the trace of the matrix A.

(i) $\lambda(A) = \sum_{i=1}^{n} \lambda_i$,

(ii) $Tr(A) = \sum_{i=1}^{n} a_{ii}$,

(iii) $\gamma(A) = \sum_{i=1}^{n} \langle Ae_i , e_i \rangle$,

where Tr(A) is called the matrix trace or the classical trace of the matrix A, $\gamma(A)$ is an another formula for the classical trace of the matrix A, and $e_i = (0, \ldots, 0, \underset{i\text{-th coordinate}}{1}, 0, \ldots, 0)$.

Proof.

We prove :(i) = (ii) = (iii).

(i) = (ii) : In the expansion (1) we must show that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = a_{11} + a_{22} + \ldots + a_{nn} . \tag{3}$$

Let $B = \lambda I - A$. In the expansion

$$\det B = \sum_{(j)} \varepsilon_{j_1 j_2 \dots j_n} b_{1j_1} b_{2j_2} \dots b_{nj_n} \tag{4}$$

$$= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_n \qquad \text{[by (1)]}.$$

The symbol (j) denotes that the summation is to be extended only over the n! permutations $j_1\, j_2\, \dots\, j_n$ of 1, 2, ..., n, and

$$\varepsilon_{j_1 j_2 \dots j_n} = \begin{cases} 1 & \text{If } j_1\, j_2 \dots\, j_n \text{ is an even permutation} \\ -1 & \text{If } j_1\, j_2 \dots\, j_n \text{ is an odd permutation.} \end{cases}$$

We shall find the coefficient c_1 of λ^{n-1}. The only permutation which contains at least n-1 factors λ is the permutation

$$j_1=1,\ j_2=2,\ \dots,\ j_n=n.$$

[In fact, for some permutation if $j_i \neq i$ for one i, then $j_i \neq i$ for at least two i ,which implies that $\prod_i b_{ij}$ has degree $\le (n-2)$ in λ].

Hence, the coefficient c_1 of λ^{n-1} can be calculated from the single term

$$b_{11}\, b_{22} \dots b_{nn} = (\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn}) . \tag{5}$$

Therefore,

$$c_1 = -a_{11} - a_{22} - \dots - a_{nn} .$$

But the identity (1) yields

$$c_1 = -\lambda_1 - \lambda_2 - \lambda_3 - \dots - \lambda_n .$$

Then

$$\lambda(A) = a_{11} + a_{22} + \dots + a_{nn} .$$

(ii) = (iii) : Since

$$Ax = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \xi_j \right) e_i , \quad \forall \; x = (\xi_j)_{j=1}^{n} .$$

Then

$$Ae_j = \sum_{i=1}^{n} a_{ij} e_i .$$

Therefore

$$\sum_{j=1}^{n} \langle Ae_j , e_j \rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \langle e_i , e_j \rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \delta_{ij}$$

i.e.

$$\sum_{j=1}^{n} \langle Ae_j , e_j \rangle = \sum_{j=1}^{n} a_{jj} = Tr(A).$$

This completes the proof. #

1.1.4. Lemma [5]

Let $A=[a_{ij}]$, $B=[b_{ij}]$ be two n×n matrices, λ be a scalar. Then

(i) $Tr(A)=Tr(A')$, where $A'=[a_{ji}]$ is the transpose of A.

(ii) $Tr(I_n) = n$, where I_n is the identity matrix of order n.

(iii) $Tr(A + B)=Tr(A) + Tr(B)$.

(iv) $Tr(\lambda A)=\lambda Tr(A)$.

Proof

We only prove (iii) and (iv).

(iii) Clearly ,

$$Tr(A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii})$$