BOUNDARY VALUE PROBLEMS
IN
NON-SMOOTH DOMAINS

A
THESIS

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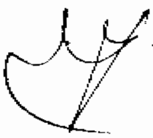 Prof. Dr. M. A. Khidr	Faculty of Science Ain Shams University
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بسم الله الرحمن الرحيم

الله

بسم الله الرحمن الرحيم

وقل رب زدني علما

صدق الله العظيم

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Atta Abu Hany

To

My Mother

and

Prof. Mohamed I. Hower

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ABSTRACT

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In the last few years, the theory of Partial Differential Equations has continued to develop rapidly.

This thesis is concerned, in a unified manner, with some of the main directions now taken in the field of Boundary Value Problems. We shall confine ourself to studying certain elliptic and parabolic problems, mainly to studying quantitative characteristics of classical and generalized solutions of these problems.

During the course of studying classical and weak solutions and their regularities, it has been noted that boundary problems for elliptic or parabolic equations may have classical solutions, which are not weak ones. This situation illustrates the fact that the recent definition of a weak solution is not a complete generalization of that of a classical one.

To avoid such situation and to construct a reasonable theory of weak solutions, there are two different approaches:

ABSTRACT :]

(I) *To* investigate the conditions that must be imposed on the known functions of the problem, i.e., the coefficients of the differential operator, the right-hand side function and the functions defining the initial and boundary conditions in order that the classical solution be in one of the Sobolev spaces H^m, and, consequently, is a weak solution of the same problem [4], [5], [9], [10], [11].

(II) *To* construct a more general concept of weak solutions, such that it generalizes the concept of classical solutions.

The latter approach was firstly introduced by Mikhailov [14] for Dirichlet's problem for a second-order elliptic equation. His idea was based on considering solutions in local Sobolev spaces, (cf; H^1_{loc}), where the boundary conditions are satisfied in the square mean.

This thesis deals with both directions mainly for the first mixed problem for a parabolic equation.

The thesis consists of four chapters. In chapter (I) we introduce the basic definitions and notations being used throughout this work. Also we include this chapter with a full and clear presentation of Sobolev spaces by which we formulate and prove our results.

Chapter (II) exposes a survey of the recent works of Mikhailov [14], [15], [16] and Hassan [8], which concerns the solution in

ABSTRACT :]

H^1_{loc} of Dirichlet's problem for an elliptic equation: its existence, uniqueness and regularity. Also, the conditions sufficient for such solution to have a limit in the square mean on the boundary are investigated.

In chapter (III) we study the integrability properties of classical solutions of a parabolic problem and their derivatives near the base of the cylindrical region, where the problem is considered. The nature of the singularities that the solution and its derivatives may have near the base are investigated. The material of this chapter has been accepted for publication in Ain Shams Science Bulletin, 1991.

In chapter (IV) we introduce a more general concept of weak solutions for a parabolic problem in local Sobolev spaces of functions of x and t. Such solutions are required to satisfy both boundary and initial conditions in the square mean. With the aid of certain eigenvalue parabolic problem we prove the existence and uniqueness of such solutions. In addition to that, a regularity result is obtained for these solutions. The material of this chapter is accepted for publication in Ain Shams University Engineering Bulletin, 1991.

CHAPTER (I)

INTRODUCTORY TOPICS

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§ 1.1 INTRODUCTION:

The process of developing Partial Differential Equations (P.D.E,s) requires a good knowledge of Sobolev spaces. In fact these spaces are an essential ingredient of any serious study of partial differential equations. Sobolev spaces are much appropriate than the spaces $C^m(\Omega)$; known as the class of m times continuously differentiable functions on Ω for any non-negative integer m. This argument counts on the fact that $C^m(\Omega)$-spaces suffer from the drawback that, with the exception of $C(\Omega)$ with the sup-norm, they are not complete.

The Sobolev spaces $H^m(\Omega)$ provide, as we shall see, a very natural setting for boundary value problems since they are complete. Furthermore, it is possible to obtain quite general results regarding the existence and uniqueness of solutions of P.D.E,s , by using these spaces. Also it provide a means of characterizing the degree of smoothness of functions. We point out here that completeness of a space depends crucially on the choice of norm.

In this chapter we shall give a brief survey of the fundamental results about the regularity problems for classical and generalized solutions.

§ 1.2 NOTATIONS:

The following notations and conventions will be used.

R^n = the real n-space. For any point $x = (x_1,\dots,x_n)$ and $y = (y_1, \dots, y_n) \in R^n$,

$$|x| = (x_1^2 + \dots + x_n^2)^{1/2}, \qquad xy = (x_1y_1 + \dots + x_ny_n) ,$$

while $(x,t) = (x_1,\dots,x_n,t)$ is a variable point of the space R^{n+1}, where t denotes the time.

Z_+^n = The set of all ordered n-tuples of non-negative integers. For any multi-index $\alpha \in Z_+^n$,

$$\alpha = (\alpha_1, \dots , \alpha_n), \qquad \alpha! = \alpha_1! \dots \alpha_n! .$$

$|\alpha|$ denotes the length of the n-tuples α and we write

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n .$$

Denoting $\partial/\partial x_i$, $i = 1,2,\dots,n$ by D_i we define the partial derivative D^α of order $|\alpha|$ by:

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} .$$

Supp u = the support of u, that is the closure of the set$\{x: u(x) \neq 0\}$.