ON THE HOMOLOGY AND COHOMOLOGY OF FIBRE BUNDLES

THESIS

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By

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"FIRST AND FOREMOST, THANKS ARE TO GOD, THE MOST BENEFICENT AND MERCIFIL"

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M. Sc. Courses

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1.	Homology Theory	2 hours weekly for one year.
2.	Functional Analysis	2 hours weekly for one year.
3.	Topology	2 hours weekly for one year.
4.	Modern Algebra	2 hours weekly for one year.
5.	Differential Equations	2 hours weekly for one year.

CONTENTS

SUMMARY

The notation of a fibre bundle first arose out of questions posed in 1930 on the topology and geometry of manifolds, |7|. By the year 1950, the definition of fibre bundle has been clearly formulated. Steenrod's book |31|, which appeared in 1950, gave a coherent treatment of the subject up to that time.

The problems connected with bundles are of various types. The problem of the great interest is that of obtaining the relations connecting the homology and cohomology structures of the total space, base space and the fibre. This problem is one of the important problems in present day in algebraic topology. There are many techniques that can be applied to that problem. Many of mathematicians have investigated this problem, we mention among them W.H.Cockroft |5|,|4|, Malm |23|,|24|, Massy |25|, M.Ginsburg |15|, Spanier |30| and H.C.Wang |32|.

In the present work, the existance of an isomorphism between the homology [cohomology] of the total space and that of the product bundle is of prime importance. This problem has also studied by many mathematicians, such as K. Drechsler |11|, A. Borel |3|, Spanier |30| and F. Abdel Halim |1|,|2|.
In general this isomorphism is not valid, |1|, although, for certain fibre bundles, the existance of such isomorphism is possible, e.g., when either the fibre is totally non-homologous to zero or a cohomology extension exists.

A.Borel, |3|, has proved the existance of the cohomology isomorphism in the first case, while Spanier, |30|, has obtained the homology and cohomology isomorphism in the second case.

In the homology case, F.Abdel Halim in |1| has introduced the corresponding notations. She defined the homology extension and the fibre is totally non-homologous to zero in the homology wise. Moreover, the homology isomorphism has been obtained in |1| if the fibre is totally non-homologous to zero in the homology wise.
On the other hand, K.Drechsler, |11|, has proposed another technique to prove the homology isomorphism in two cases of fibre bundle, in these two cases the homology of the fibre and base spaces are considered to be free. Moreover, in the first case some assumptions are considered on the base and the total spaces, while in the second case another assumptions are considered on the base space only. These results are proved under one of the following two conditions:

(I) The category of spaces is the category of compact Hausdorff regular pairs, and the homology is an arbitrary.

(II) The category under consideration is the category of compact Hausdorff pairs, and the homology is continuous.

Using the Drechsler's technique, F. Abdel Halim has proved, |1|,|2| the existance of the homology and cohomology isomorphisms for a fibre bundle whose spaces satisfies some retraction properties. Moreover, under condition (I), the dual result to the first case of the Drechsler's results is given in |2|.

In the present thesis, we prove the dual result to the second one of Drechsler's results under any one of conditions (I) and (II). We show that the results given in |2| is also valid under condition (II). Moreover, we will prove that the cause of the validity of the homology [cohomology] isomorphism is that the fibre is totally non-homologous to zero in the homology [cohomology] wise.

The thesis consists of three chapters.

The first chapter contains basic ideas from algebra, topology and algebraic topology. These ideas form the theoretical base of our study.

The second chapter deals with the answer of the question "Are relative homeomorphic spaces have isomorphic homology [cohomology] groups ?". In the first article of chapter two a brief account about continuous homology and cohomology is given. The second article provides with the result of K. Drechsler, |9|, in the homology case under any one of two condition (I),(II). In the cohomology case, F.Abdle Halim, |2|, has proved the dual result under condition (I). In the present work, we prove, under condition (II), the dual result in the cohomology case.

The third chapter is devoted to discuss the answer of the question "Is the homology [cohomology] of the total space of a fibre bundle isomorphic to that of the product bundle?". The first article of chapter three is a summary of the previous results deals with this question. In the second article we prove the main results of our study.

CHAPTER I

CHAPTER (I)

BASIC CONCEPTS

In this chapter we introduce the definitions and conventions which are required for our study. We classify them into six main sections. In the first section we consider the topological concepts. The second section is devoted to the algebraic concepts. The third section deals with the direct and inverse systems. Basic facts about category and functors are presented in section four. The axioms for homology and cohomology are considered in section five. Finally, we concern with the singular homology theory in the last section.

(1.1) Topological Concepts

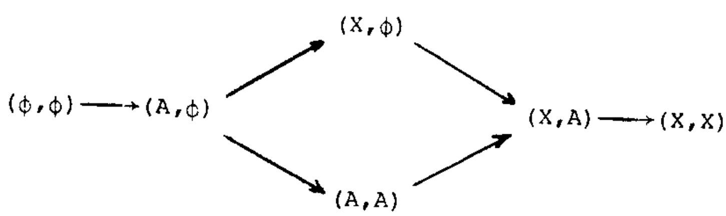
This section deals with some topological constructions and some special continuous maps. Also, a brief account on the topology of fibre bundle will be given.

Definition (1.1.1). A topological pair (X,A) consists of a topological space X, and a subspace A of X. If A is empty, we shall not distinguish between the pair (X,ϕ) and the space X.

By a map $f:(X,A) \longrightarrow (Y,B)$ from a topological pair (X,A) into a topological pair (Y,B), we mean a (continuous) map $f:X \longrightarrow Y$ from the topological space X into the topological space Y satisfying $f(A) \subset B$. We shall not distinguish between the map $f:(X,\phi) \longrightarrow (Y,\phi)$ and the map $f:X \longrightarrow Y$, |13|.

Definition (1.1.2). A topological triple (X,A,B) is a topological space X together with two subspaces A and B of X satisfying $B \subset A$, |20|.

Definition (1.1.3). The lattice of a pair (X,A) consists of the pairs:



all their identity maps, the inclusion maps indicated by arrows, and all their compositions.

If $f:(X,A) \longrightarrow (Y,B)$, then f defines a map of every pair of the lattice of (X,A) into the corresponding pair of the lattice of (Y,B), |13|.

Definition (1.1.4). Let $\{X_\alpha\}$ be a collection of sets indexed by a set M. The product $\prod_{\alpha \in M} X_\alpha$ of the collection $\{X_\alpha\}$ is the set of functions $x = \{x_\alpha\}$ defined for each $\alpha \in M$ and such that x_α, the value of x on α, is an element of X_α. In particular, if $M = \{1,2,\ldots,n\}$, then the product $\prod_{\alpha \in M} X_\alpha$ is the set of all n-tuples $(x_1,x_2,\ldots,x_n)$ such that $x_i \in X_i$, $i=1,2,\ldots,n$, and will be denoted by $X_1 \times X_2 \times \ldots \times X_n$. If each X_α is a topological space, a topology is introduced in the product of the collection $\{X_\alpha\}$ as follows:

If a finite number of X_α's are replaced by open subsets $U_\alpha \subset X_\alpha$, the product of the resulting collection is a subset of $\prod_{\alpha \in M} X_\alpha$, and is called a rectangular open set of $\prod_{\alpha \in M} X_\alpha$. Any union of rectangular open sets is called an open set of the product. The product with this topology is called the topological product, and the space X_α is called α-component of the product space, |13|.

Definition (1.1.5). Let $F = \{X_\alpha, \alpha \in M\}$ be an indexed family of spaces, and let $X = \bigcup_{\alpha \in M} X_\alpha$ be the union of all sets X_α. Define a collection $\mathcal{T}$ of subsets of X as follows; A subset U of X is in $\mathcal{T}$ iff $U \cap X_\alpha$ is an open set of the space X_α for every $\alpha \in M$. It is obvious that $\mathcal{T}$ is a topology on X. The set X with this topology is called the topological sum or sum of the family F, |18|.

For each $\alpha \in M$, let $\{\alpha\}$ be the singleton space and X'_α denotes the topological product $\{\alpha\} \times X_\alpha$. The topological sum of the disjoint family $F' = \{X'_\alpha, \alpha \in M\}$ is called the disjoint topological sum of F, |18|.

Definition (1.1.6). Let $f:X \longrightarrow Y$ be a surjective function from a space X onto a set Y. The identification topology on Y determined by f is the topology in which $U \subseteq Y$ is closed if $f^{-1}(U)$ is closed in X, |17|.

Definition (1.1.7). Let R be an equivalence relation on the points of a space X. Let Y be the set of R-equivalence classes, the function $k:X \longrightarrow Y$ which assigns to each point $x \in X$ its equivalence class is called the natural projection of X into Y. If Y is given the identification topology determined by R, we write $Y=X/R$ and say that Y is the quotient space of X by the relation R, |17|.

<u>Definition (1.1.8).</u> Let $f:X \longrightarrow Y$ be a map and let $X \times I$ be the topological product of X and the closed unit interval $I=[0,1]$. Consider the disjoint topological sum $T = (X \times I) + Y$. Introduce into T an equivalence relation generated by the rule $(x,0)R\, f(x)$ for all $x \in X$. Then the quotient space of T by R is called the mapping cylinder of f, and will be denoted by Y_f, |17|.

<u>Definition (1.1.9).</u> Consider an arbitrary topological pair (X,A). An open neighbourhood V of A in X is said to be regular if there exists a map

$$\Phi : b(V) \longrightarrow A$$

from the boundary $b(V) = cl(V) - V$ of V into A such that the closure cl(V) of V in the space X is homeomorphic to the mapping cylinder of the map Φ.

A topological pair (X,A) is said to be regular if $A \neq \phi$ and there exists a regular open neighbourhood V of A in the space X, |20|.

<u>Definition (1.1.10).</u> Consider any two continuous maps

$$f,g: (X,A) \longrightarrow (Y,B)$$

then f and g are homotopic if there exists a map

$$h: (X,A) \times I \longrightarrow (Y,B)$$

such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. The map h is called a homotopy from f to g, |13|, |30|.

<u>Definition (1.1.11).</u> A subspace A of space X is said to be a retract of X if there exists a continuous map $r:X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$.
Such a map r is called a retraction of X onto A, |20|.

<u>Definition (1.1.12).</u> A continuous map $f:(X,A) \longrightarrow (Y,B)$ is called a relative homeomorphism if f maps X-A homeomorphically onto Y-B, |13|.

<u>Definition (1.1.13).</u> A set G is called a topological group if G is a group, G is a topological space and the group operations in G are continuous. More precisely, asserts that the transformations $G \times G \longrightarrow G$, $G \longrightarrow G$ given by $(x,y) \longrightarrow xy$, $x \longrightarrow x^{-1}$ are continuous, |17|.

The notion of a fibre bundle first arose out of questions posed in 1930, on the topology and geometry of manifold. By the year 1950 the definition of fibre bundle has been clearly formulated |7|.

In the literature various notions of fibration has been considered, but Hürerwize has led to the correct notion, |30|. In |30| it is proved that the bundle projection of fibre bundle is a fibration.

<u>Definition (1.1.14).</u> A fibre bundle (E,B,F,P) consists of a total space E, a base space B, a fibre space F and a bundle projection $P:E \longrightarrow B$ such that there exists an open covering $\{U_m\}_{m \in M}$ of B and , for each $U_m \in \{U_m\}$, a homeomorphism $\Phi_m:U_m \times F \longrightarrow P^{-1}(U_m)$

such that the composition

$$U_m \times F \xrightarrow{\Phi_m} P^{-1}(U_m) \xrightarrow{P} U_m \qquad (1.1.1)$$

is the projection to the first factor. Thus the bundle projection $P:E \longrightarrow B$ and the projection $B \times F \longrightarrow B$ are locally equivalent. The fibre over $b \in B$ is defined to equal $P^{-1}(b)$ and will be denoted by F_b. Note that F is homeomorphic to F_b for every $b \in B$, |30|. This homeomorphism will be denoted by $\phi_b : F_b \simeq F$.

An example of fibre bundle is the product bundle or product space $E = B \times F$. In this case, the projection is given by $P(x,y)=x$. Taking $U_m = B$ and Φ_m = the identity, condition (1.1.1) is fulfilled.

An important problem in algebraic topology, called homotopy lifting problem, is defined in the following definition.

<u>Defintion (1.1.15).</u> A map $P: E \longrightarrow B$ is said to have the homotopy lifting property with respect to space X if, given maps $f': X \longrightarrow E$ and $F:X \times I \longrightarrow B$ such that $F(x,0) = Pf'(x)$ for $x \in X$, there is a map $F': X \times I \longrightarrow E$ such that $F'(x,0) = f'(x)$ for each $x \in X$ and $P \circ F' = F$. If f' regarded as a map of $X \times 0$ to E, the existance of F' is equivalent to the existance of a map represented by the dotted arrow that makes the following diagram commutative, |30|: