THE PROJECTIVE COHOMOLOGY GROUPS

THESIS

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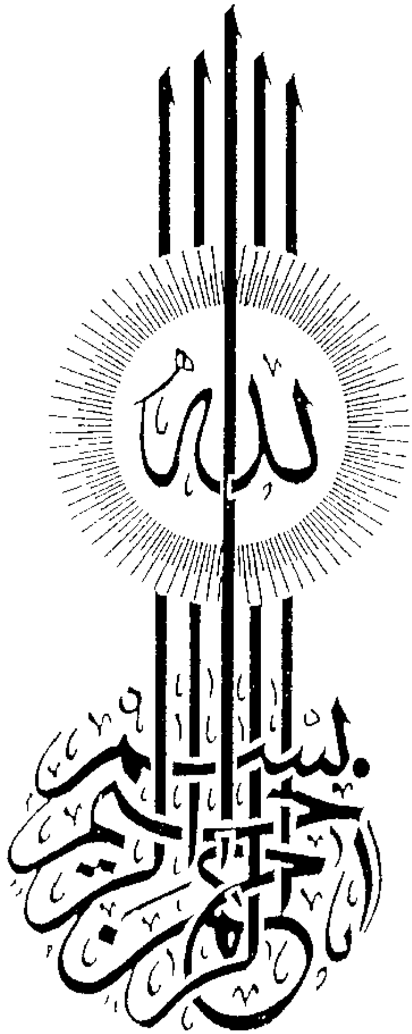
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"FIRST AND FOREMOST, THANKS ARE TO GOD,

THE MOST BENEFICENT AND MERCIFUL".

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PREFACE

P R E F A C E

The cohomology theory is one of the most important subjects in algebraic topology, its axiomatic characterization has been given firstly by S. Eilenberg and N. Steenrod, [13], as that for the homology theory. Its importance is due to the fact that an algebraic representation of topology converts topological problems into algebraic ones, to the end that with sufficiently many representations, the topological problems will be solvable if (and only if) all the corresponding algebraic problems are solvable.

Many definitions have been given for extending the cohomology theory of complexes to arbitrary topological spaces. At the present the two theories most commonly are the Alexandrov-Čech, [11], [13] and the singular theories, [12],[13]. It is known that there exist compact Hausdorff spaces on which these two theories differ. A different definition was given in 1935 by J.W. Alexander, [5], who constructed cohomology groups for compact metric spaces using functions of sets of points in the space. These groups with suitable choice of coefficient groups, are isomorphic to the Alexandrov-Čech Cohomology groups. Inasmuch as the definition involving functions is simpler than the definition in terms of covering, usually used to obtain the Alexandrov-Čech groups, a generalization of Alexander's definition, [5], applicable to arbitrary space is desirable. E.H. Spanier, in [22] has suggested such a definition which is valid for any space and is even somewhat simpler than Alexander's definition. On compact Hausdorff spaces, these groups agree with the Alexandrov-Čech Cohomology groups. However, the most widespread cohomology theory applicable in sufficiently wide categories of topological spaces is the Alexandrov-Čech Cohomology theory (Spectral Cohomology).

It is known that the concept of a group spectrum and its limit group yields a particularly elegant method for defining the Alexandrov-Čech homology and cohomology groups. It has shown in [16] that these fundamental group-theoretic notions can also be applied to define the connectivity groups, in the sense of "singular" homology theory. With this approach, the "singular" homology groups appear as direct limits and the "singular" cohomology groups are inverse limits, in exact contrast with Alexandrov-Čech theory. In view of the direct limit concept it is introduced in [1], the idea of the projective homology groups of compact spaces which is a generalization of the interesting homology groups of Steenrod, [24]. In [3], it is discussed the generalization of the concept of the direct limit groups , [16].

CHAPTER I

Using this idea projective homology construction over a chain complex is introduced and studied in [4].

In the present thesis we give some topics of spectrums with morphisms which are analogous to the material of inverse systems. Also we give a construction of the projective cohomology groups of a pair of compact Hausdorff spaces over a chain complex as a coefficient group. Moreover, this construction is discussed from the point of view of the first four axioms of the Eilenberg-Steenrod axioms, [13].

The thesis consists of three chapters :

The first chapter contains basic ideas from topology, algebra and algebraic topology. These ideas form the theoretical base of our study.

The second chapter, seems to be original, deals with a slight generalization of the concept of inverse systems to include the case when there may be more than one morphism between two objects of the spectrum.

In **the third chapter**, which represents the main original effort in the thesis, the projective cohomology construction over a chain complex is introduced, and it is proved that this construction is δ -functor.

CHAPTER I

CHAPTER 1

SOME BASIC CONCEPTS AND CONVENTIONS

In this chapter we introduce the definitions and the conventions which are required for our study in this thesis. We classify them into six main sections. In the first section we consider a general topological and algebraic concepts. The second section deals with the idea of the categories and functors. The third section deals with the axioms for cohomology theory. In the fourth and fifth sections we elaborate cochain complexes , simplicial complexes and their cochain complexes which lead to the construction of their cohomology groups. Finally we concern with the duality theory in the last section.

(1.1) Algebraic and Topological Concepts.

Definition (1.1.1): A lower sequence of groups is a collection $G = \{G_q , \Phi_q\}$, where for each integer q(positive, negative, or zero), G_q is a group, and $\Phi_q : G_q \longrightarrow G_{q-1}$ is a homomorphism.

The lower sequence G is said to be exact if, for each integer q , $Im\, \Phi_{q+1} = \ker \Phi_q$, where

$Im\, \Phi_{q+1} = \Phi_{q+1}(G_{q+1})$ and $\ker \Phi_q = \{g \in G_q \mid \Phi_q(g)=0\}$, [13].

Definition (1.1.2): An upper sequence of groups is a collection $G = \{G^q , \Phi^q\}$, where for each integer

q(positive, negative, or zero), G^q is a group and
$\phi^q : G^q \longrightarrow G^{q+1}$ is a homomorphism.
An upper sequence is exact if, for each integer q,
$I_m \ \phi^{q-1} = \text{Ker} \ \phi^q$.

Definition (1.1.3): A lower [upper] sequence $\{G_q , \phi_q\}$ is said to be of order 2 or semi-exact if the composition of any two successive homomorphisms of the sequence is zero, i.e., for each q , $\phi_q \phi_{q+1} = 0$. This means that $\text{Im}\ \phi_{q+1}$ is a subgroup of $\text{Ker}\ \phi_q$ for any q.

Remark (1.1.1): There is a one-to-one correspondence between the set of all lower sequences and the set of all upper sequences,for, if $G = \{G_q, \phi_q\}$ is a lower sequence and we set $G^q = G_{-q}$, and $\phi^q = \phi_{-q}$ for each q, then $\{G^q , \phi^q\}$ is an upper sequence .In the sequal definitions are made only for upper sequences. The corresponding definitions for lower sequences are to be obtained using this transformation, [13].

Definition (1.1.4): An upper sequence $G' = \{G'^q , \phi'^q\}$ is said to be a subsequence of the upper sequence $G = \{G^q, \phi^q\}$, written $G' \subset G$, if, for each integer q, G'^q is a subgroup of G^q and the homomorphism ϕ'^q is the restriction of ϕ^q on G'^q .

Definition (1.1.5): If $G = \{G^q, \phi^q\}$ and $G' = \{G'^q, \phi'^q\}$ are two upper sequences, a homomorphism $\psi : G \longrightarrow G'$ is a sequence of homomorphisms, $\{\psi^q\}$ such that, for each integer q, $\psi^q : G^q \longrightarrow G'^q$, and the following diagram is commutative;

$$\begin{array}{ccc} G^q & \xrightarrow{\ \phi^q\ } & G^{q+1} \\ {\scriptstyle \psi^q}\downarrow & & \downarrow{\scriptstyle \psi^{q+1}} \\ G'^q & \xrightarrow{\ \phi'^q\ } & G'^{q+1} \end{array}$$

i.e., $\psi^{q+1} \phi^q = \phi'^q \psi^q, \quad \forall q$.

If each ψ^q is an isomorphism, ψ is said to be an isomorphism.

Definition (1.1.6): If $L = \{L^q, \psi^q\}$ is a subsequnce of the upper sequence $G = \{G^q, \phi^q\}$, the factor sequence G/L of G by L is the upper sequnce composed of the factor groups G^q/L^q, and the homomorphisms $\tilde{\phi}^q : G^q/_{L^q} \longrightarrow G^{q+1}/_{L^{q+1}}$ which are induced by the ϕ^q .

Definition (1.1.7): Let $\{X^\alpha\}$ be a collection of sets indexed by a set M, i.e, for each $\alpha \in M$, X^α is a set of the collection. The product of the collection, denoted by $\prod_{\alpha \in M} X^\alpha$, is the totality of functions $x = \{x^\alpha\}$

defined for every $\alpha \in M$ and such that x^{α}, the value of x on α is an element of X^{α}. The element x^{α} is called the α-coordinate of x. In particular, if M consists of the first n natural numbers $1,2,\ldots,n$, then the product of the collection is the set of all n-tuples $(x^1, x^2,\ldots, x^n)$ such that $x^i \in X^i$, $i = 1,2,\ldots,n$ and will be denoted by $X^1 \times X^2 \times \ldots \times X^n$.

If each X^{α} is a topological space, a topology is introduced in the product of the collection $\{X^{\alpha}\}$ as follows :

If a finite number of the X^{α}'s are replaced by open subsets $U^{\alpha} \subset X^{\alpha}$, the product of the resulting collection is a subset of $\prod_{\alpha \in M} X^{\alpha}$, and is called a rectangular open set of $\prod_{\alpha \in M} X^{\alpha}$.

Any union of rectangular open sets, is called an open set of the product. The product with this topology is called the topological product.

If each X^{α} in the collection $\{X^{\alpha}\}$ is an abelian group, then an addition is defined in the product $\prod_{\alpha \in M} X^{\alpha}$ by the usual method of adding function values :

$$(x + x')(\alpha) = x(\alpha) + x'(\alpha) = x^{\alpha} + x'^{\alpha}$$

In this way the product $\prod_{\alpha \in M} X^{\alpha}$ becomes an abelian group and is called the direct product of the groups $\{X^{\alpha}\}$, [13], [26].