COMPOSITION OPERATORS

A THESIS

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By

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الله

بسم الله الرحمن الرحيم

وقل اعملوا فسيرى الله عملكم ورسوله والمؤمنون

« صدق الله العظيم »

بسم الله الرحمن الرحيم

M. SC. COURSES

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FACULTY OF SCIENCE

1. Algebraic topology.	2 hours per week (during the whole accademic year)
2. Functional analysis.	2 hours per week
3. Linear algebra.	2 hours per week
4. Logic and graph theory.	2 hours per week
5. Modern algebra.	2 hours per week (for one semister)
6. Differential geometry.	2 hours per week (for one semister).

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PREFACE

The theory of linear operators on Hilbert spaces is one of the active parts of functional analysis.

In 1977, a general conference was held at California State University Long Beach to discuss Hilbert space operators. The purpose of this conference was to put a classification to these operators as the following:

(1) Integral Operators $(f(x) \to (\int k(x,y)f(y)dy)$.

(2) Multiplication Operators $(f(x) \to \phi(x)f(x))$.

(3) Composition Operators $(f(x) \to f(\phi(x)))$.

Our goal in this research is to study one of these operators, namely composition operators which are discussed in one of the main papers introduced in this conference by Eric A. Nordgren.

To define a composition operator, suppose V is a vector space of complex valued functions on a set X under the pointwise operations of addition and scalar multiplication. If ϕ is a mapping of X into X such that the composite $f\circ\phi$ of f with ϕ is in V whenever f is , then ϕ induces a linear transformation C_ϕ on V that sends f into $f\circ\phi$. This linear transformation C_ϕ is called a composition operator on V if C_ϕ is a bounded and V is a Hilbert space.

There are many contexts in which such a study can be carried out. Our principal work has been devoted to the case in which the Hilbert space V is the L^2 space of a measure.

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There are many contexts in which such a study can be carried out. Our principal work has been devoted to the case in which the Hilbert space V is the L^2 space of a measure.

This thesis consists of three chapters.

In chapter I, we state some basic definitions, notations and theorems which we need in this thesis. We give more attention for the basic concepts, results and important theorems on measure theory which represents a main material for building most parts of our thesis.

In chapter II, we characterize composition operator using the Nordgren's conditions (Th. 2.2.1).

Our work in this chapter is to show that the condition of Singh (Th. 2.3.1) for characterization of composition operator, is equivalent to Nordgren's conditions. Also, we give necessary and sufficient conditions for the set of all composition operators induced by all one-to-one measurable transformations of X onto X to form a group under composition of mappings (§2.4).

Characterization of composition operators induced by monotone functions, rational functions and polynomials on the Real line with Lebesgue method will be investigated at the end of this chapter.

In chapter III, we study some special types of composition operators on the L^2 space of a measure. Conditions for a composition operator to be normal (Th. 3.1) and some related results are given.

Some conditions under which the composition operator is unitary and Hermitian are investigated, and some examples of unitary composition operators on the L^2 space of a special measure are solved.

Finally, some results concerning compact composition operators on the L^2 space of a measure are studied.

In this thesis, we have represented a group of transformations on some measure theory as a group of transformations on L^2 spaces , also some conditions on ϕ to make C_ϕ normal operator are obtained .

CONTENTS

CHAPTER I

GENERAL INTRODUCTION

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Most of this chapter consists of certain basic definitions and results which will be required later.

§(1.1) Some Basic Concepts of Measure Theory

(1.1.1) Definition (Measurable Space) [6]

(a) A collection δ of subsets of a set X is said to be a σ-algebra (sigma algebra) in X, if δ has the following properties:

(i) $X \varepsilon \delta$

(ii) If $E \varepsilon \delta$, then $E^c \varepsilon \delta$, where E^c is the complement of E relative to X.

(iii) If $E = \bigcup_{n=1}^{\infty} E_n$ and if $E_n \varepsilon \delta$ for $n = 1,2,3,\ldots$, then $E \varepsilon \delta$.

(b) If δ is a σ-algebra in X, then X is called a measurable space, and the members of δ are called the measureable sets in X.

(c) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

(1.1.2) Remarks: [6] & [3]

(1) Let δ be a σ-algebra in a set X. Reffering to Properties (i) to (iii) of Definition (1.1.1)(a), we immediately derive the following facts:

(a) Since $\Phi = X^c$, (i) and (ii) imply that $\Phi \varepsilon \delta$ (where Φ is the empty set).

(b) Taking $E_{n+1}=E_{n+2}=\ldots=\Phi$ in (iii), we see that

$$E_1 \cup E_2 \cup \ldots \cup E_n \epsilon \delta \quad \text{if } E_i \epsilon \delta \quad \text{for } i=1,2,\ldots,n .$$

(c) Since $\bigcap_{n=1}^{\infty} E_n = (\bigcup_{n=1}^{\infty} E_n^c)^c$, δ is closed under the formation of countable (and also finite) intersections.

(d) Since $E-F = F^c \cap E$, we have

$$E-F\epsilon\delta \quad \text{if} \quad E\epsilon\delta \quad \text{and} \quad F\epsilon\delta.$$

The prefix σ refers to the fact that (iii) is required to hold for all countable unions of members of δ. If (iii) is required for finite unions only, then δ is called an algebra of sets.

(2) A class of sets, R, is called a ring if whenever $E\epsilon R$ and $F\epsilon R$ then $E \cup F$ and $E-F$ belong to R.

A ring is called a σ-ring if it is closed under the formation of countable unions.

It is possible to show that every algebra is a ring and every σ-algebra is a σ-ring. In other words, an algebra may be characterized as a ring containing X, and σ-algebra is a σ-ring containing X.

(3) There exists a smallest ring and a smallest σ-ring containing a given class of subsets; we refer to these as the generated ring and generated σ-ring respectively.

(4) A non empty class M of sets is monotone if, for every monotone sequence (E_n) of sets in M, we have

$$\lim_{n} E_n \in M .$$

A σ-ring is a monotone; a monotone ring is a σ-ring . Also, if a monotone class contains a ring R then it contains the smallest σ-ring containing R.

(1.1.3) Propositions: [6]

(a) If $f=u+iv$, where u and v are real measurable functions on the measurable space X, then f is a complex measurable function on X.

(b) If E is a measurable set in X, and if

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

then χ_E is a measurable function.

We call χ_E the characteristic function of the set E.

The letter χ will be reserved for characteristic functions throughout this thesis.

Note: For any two measurable sets E and F in X, we have:

(1) $\chi_{E \cap F} = \chi_E \cap \chi_F = \chi_E \circ \chi_F = \chi_E \chi_F$.

(2) $\chi_{E \cup F} = \chi_E \cup \chi_F = \chi_E + \chi_F - \chi_E \chi_F$.

(3) $\chi_{E^c} = 1 - \chi_E$ (see [11] & [5]).

(1.1.4) Definition (Borel sets) [6]

Let X be a topological space. By the third remark of (1.1.2), there exists a smallest σ-algebra β in X such that every open set in X belongs to β. The members of β are called the Borel sets of X.

In particular, closed sets are Borel sets (being, by definition, the complement of open sets), and so are all countable unions of closed sets, and all countable intersections of open sets.

Since β is a σ-algebra in X, we may regard X as a measurable space with the Borel sets playing the role of the measurable sets.

If $f:X \rightarrow Y$ is a continuous mapping of X, where Y is any topological space, then $f^{-1}(V) \varepsilon \beta$ for every open set V in Y.

In other words every continuous mapping of X is Borel measurable. Borel measurable mappings are often called Borel mappings or Borel functions.

(1.1.5) Theorem [6]

Suppose $\mathcal{S}$ is a σ-algebra in X, and Y is a topological space. Let f maps X into Y.

(a) If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \varepsilon \mathcal{S}$, then Ω is a σ-algebra in Y.

(b) If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \varepsilon \mathcal{S}$.

(c) If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \varepsilon \mathcal{S}$ for every real α, then f is measurable.