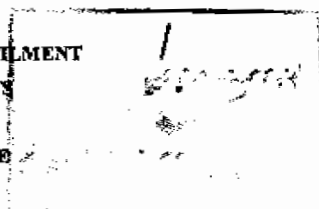


Ain Shams University



# ON TENSOR PRODUCTS OF BANACH SPACES AND SOME APPLICATIONS

THESIS  
SUBMITTED IN PARTIAL FULFILLMENT  
OF THE REQUIREMENTS  
FOR THE AWARD  
OF THE (M.Sc.) DEGREE  
IN  
(Pure Mathematics)



515.63  
S . 17

BY

52184

*Sahar Mohamed Ali*

UNDER SUPERVISION

**Dr. Nashat Faried Mohamed Fathy**

Associate Professor of Pure Mathematics  
Department of Mathematics  
Faculty of Science  
Ain Shams University



*Nashat Faried*

SUBMITTED TO  
Department of Mathematics  
Faculty of Science  
Ain Shams University  
CAIRO, EGYPT  
1995

the 1990s, the number of people with a mental health problem has increased by 50% (Mental Health Foundation 1999).

There is a growing awareness of the need to address the needs of people with mental health problems, and a number of initiatives have been developed to improve the lives of people with mental health problems (Mental Health Foundation 1999).

The purpose of this paper is to describe the development of a self-help manual for people with mental health problems, and to discuss the implications for practice.

## Background

The number of people with a mental health problem has increased by 50% in the 1990s (Mental Health Foundation 1999). This increase has been attributed to a number of factors, including an increase in the number of people with a mental health problem, and an increase in the number of people who are aware of their condition.

The increase in the number of people with a mental health problem has led to a number of initiatives to improve the lives of people with mental health problems. These initiatives include the development of self-help manuals, the provision of support groups, and the development of community mental health teams.

The purpose of this paper is to describe the development of a self-help manual for people with mental health problems, and to discuss the implications for practice.

The self-help manual was developed by a team of mental health professionals, including a psychologist, a social worker, and a nurse. The manual was designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual was developed in a number of stages. The first stage was to identify the needs of people with mental health problems. This was done by conducting a survey of people with mental health problems, and by consulting with mental health professionals.

The second stage was to develop the content of the manual. This was done by conducting a literature review, and by consulting with mental health professionals. The content of the manual was designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The third stage was to develop the format of the manual. This was done by consulting with mental health professionals, and by conducting a pilot study. The format of the manual was designed to be easy to use, and to be appealing to people with mental health problems.

The fourth stage was to evaluate the manual. This was done by conducting a pilot study, and by conducting a full-scale evaluation. The results of the pilot study and the full-scale evaluation were used to improve the manual.

The manual is now available to people with mental health problems. It is available in a number of formats, including a printed manual, a CD-ROM, and a website. The manual is available in a number of languages, including English, Spanish, and French.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.

The manual is designed to be used by people with mental health problems. It is not intended to be used by mental health professionals. The manual is designed to provide people with mental health problems with information about their condition, and to provide them with strategies for coping with their condition.



## ACKNOWLEDGEMENT

Glory be to Allah, the beneficent, the merciful whom we worship.

I wish to express my sincere and deep gratitude with many thankfulness to Dr. Nashat Faried Mohammed Fathy, Associate Professor of Pure Mathematics Ain Shams University, Faculty of Science for suggesting the topic of the thesis, for his encouragement and invaluable discussions and kind help during the preparation of the thesis.

Deep great fullness is to all my family members. My father, my mother, my sisters and my brothers. To whom I'm completely indepted.

I would like also to record my great thanks to all my Colleagues in the Mathematical Department, Faculty of Science, Ain Shams University, for encouragement and providing the necessary facilities.

1000

1

1000

1

# CONTENTS

	Page
INTRODUCTION.....	i
CHAPTER 0: SOME NOTATIONS AND BASIC DEFINITIONS	
0.1 Locally Convex Spaces.....	1
0.1.1 Bases of Neighborhoods.....	1
0.1.2 Topological Vector Spaces.....	3
0.1.3 Locally Convex Topologies on $B(X; Y)$ .....	4
0.1.4 Some Important Particular Cases of Locally Convex Topologies on the Space $B(X; Y)$ .....	7
CHAPTER 1: PROJECTIVE AND INJECTIVE TENSOR PRODUCTS	
1.1 Introduction to Tensor Products.....	11
1.1.1 Starting up using finite rank operators...	11
1.1.2 Abstract prelude to tensor products.....	12
1.1.3 Tensor products in terms of bilinear forms.	16
1.2 Projective Tensor Products.....	18
1.3 Construction of Some Topologies on The Space $\text{bil}(X * Y; H)$ .....	22
1.3.1 Construction of the Biequicontinuous Topology on the Space $\text{bil}(X_s^* * Y_s^*, H)$ .....	24
1.4 Some Fundamental Theorems for Locally Convex Spaces.....	25

## **INTRODUCTION**

## INTRODUCTION

Grothendieck's book "Produits tensoriels topologiques et espaces nucléaires" published in 1956 is one of the very few books which have deeply influenced the course of Functional Analysis. It not only demonstrated the enormous possibilities for the using of tensor products in Banach space theory, but it anticipated the study of Banach spaces in terms of finite dimensional subspaces.

Apparently few researchers noted its importance. This, in spite of the fact that Grothendieck's theory of locally convex tensor products and nuclear spaces had already won great appreciation.

Tensor products apparently appeared in Functional Analysis for the first time during the late thirties in the work of Murray and John von Neumann on Hilbert spaces. The first systematic study of classes of norms on tensor products of Banach spaces is due to R. Schatten who in 1943 continued his work in a series of papers.

The projective tensor norm  $\Pi$  and injective tensor norm  $\epsilon$  proved their usefulness rather quickly through simple examples involving vector valued function spaces. New results about general tensor norms were obtained around 1970.

Back in 1965 A. Pietsch has raised the following questions.

- 1) Must a locally convex space  $X$  be nuclear if the two spaces

$X \otimes_{\epsilon} X$  and  $X \otimes_{\Pi} X$  are homeomorphically identical?



- 2) What properties a locally convex space  $Y$  must possess in order that the projective tensor product  $X \otimes_{\pi} Y$  and the injective tensor product  $X \otimes_{\epsilon} Y$  are homeomorphically identical?
- 3) Is a locally convex space  $X$  nuclear, if all bilinear forms on  $X \times X$  are nuclear?

The answer of the first question was included in Pisier's work in Banach space theory (starting around 1975) which finally drew new attention to the tensor product approach. A highlight is certainly his solution of the problem of finding an infinite dimensional Banach spaces  $X$  for which

$$X \otimes_{\epsilon} X \neq X \otimes_{\pi} X$$

holds isomorphically.

H. Jarchow and K. John have solved the second and the third questions in 1988 and 1991 see [5, 6].

The thesis consists of six chapters

#### Chapter 0:

It contains some basic definitions and construction of some locally convex topologies that will be used.

#### Chapter 1:

Our main interests of study are the properties of the following two topologies on the tensor product  $X \otimes Y$  namely,

- 1- The strongest locally convex topology on the tensor product  $X \otimes Y$  such that the canonical bilinear mapping  $\phi$  from  $X \times Y$  into  $X \otimes Y$  defined by  $\phi(x, y) = x \otimes y$  is continuous. This topology is

known as the projective tensor topology  $\Pi$  and led us to the study of the concept of nuclear operators from  $X$  into  $Y$ .

2- The coarsest locally convex topology on the tensor product  $X \otimes Y$  such that the canonical bilinear mapping  $\phi$  (from  $X \otimes Y$  into the space of all continuous bilinear forms  $\beta_c(X_s^* \times Y_s^*)$ ) defined by  $[\phi(\sum_{i=1}^n x_i \otimes y_i)](f, g) = \sum_{i=1}^n f(x_i) g(y_i)$  is continuous. This topology is known as the injective tensor product  $\epsilon$  and led us to the study of the concept of approximable operators which is a subclass of the class of compact operators from  $X$  into  $Y$ .

## Chapter 2:

In this chapter we study some questions which raised by Banach such as: Is every compact operator approximable?. Is the canonical imbedding of  $X \hat{\otimes}_{\Pi} Y$  onto the space of all nuclear operators  $N(X; Y)$  is injective? Is the trace defined for nuclear operators?. These questions find an answer with the help of Grothendieck's approximation property.

## Chapter 3:

In this chapter we study a partial ordering mentioned in [21] among finite collections of elements of a Banach space  $X$  as follows:

$$y = \{y_i\}_{i=1}^n \leq x = \{x_i\}_{i=1}^m$$

if and only if for every  $f \in X^*$ ,

$$\left( \sum_{i=1}^n |f(y_i)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^m |f(x_i)|^2 \right)^{\frac{1}{2}}.$$

He showed that this happens if and only if  $y$  is the image of  $x$  under an  $n \times m$  matrix.

In terms of this partial ordering G. Pisier had given a necessary and sufficient condition for an operator  $T$  to be factored through a Hilbert space. During this chapter we also study the 3<sup>rd</sup> topology which is lying between the projective and injective topologies induced by the norm  $\| \cdot \|_1$  (see [21]), has a great connection with operators factoring through Hilbert spaces.

#### Chapter 4:

In this chapter we study the relations between the injective and projective tensor products, nuclear mappings between locally convex spaces and the concept of nuclear spaces.

#### Chapter 5:

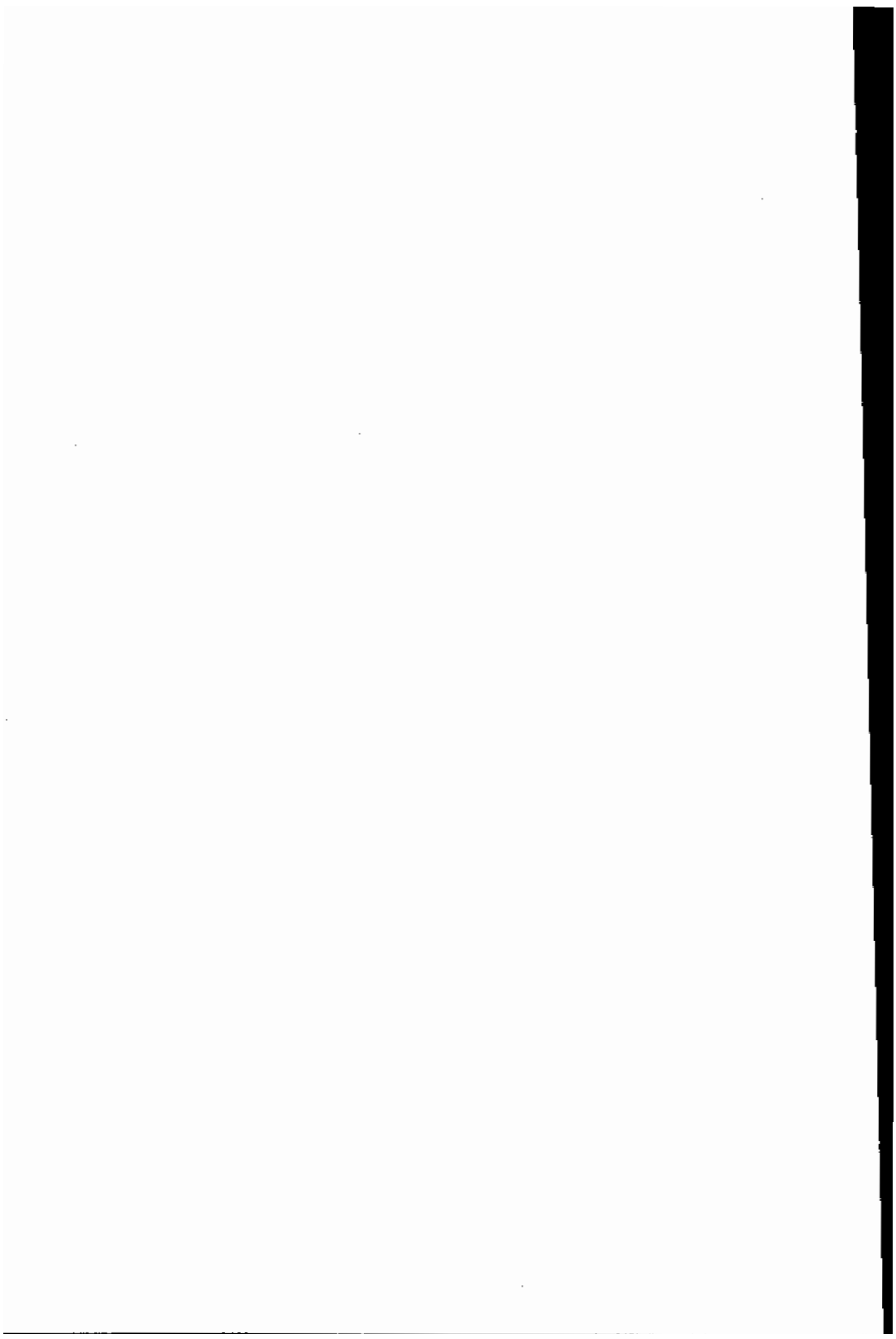
In this chapter we use infinite matrices defining bounded linear operators on the space  $c_0$  of convergent to zero sequences to define operators on the classical Banach space  $c_0(X)$  and define a partial ordering among convergent to zero sequences according to values depending on functionals and prove that this partial ordering could be obtained by application of infinite matrices.

We get an analogous result to that mentioned in [21].

For the completeness of our work we had to rewrite many proofs in a way compatible to the sequel of the thesis.

# **CHAPTER (0)**

## **SOME NOTATIONS AND BASIC DEFINITIONS**



## CHAPTER 0

### SOME NOTATIONS AND BASIC DEFINITIONS

#### 0.1 Locally Convex Spaces.

##### 0.1.1 Bases of Neighborhoods

If  $E(x)$  is the class of all neighborhoods of a point  $x$  of the topological space  $X$ , a subclass  $\beta(x)$  of  $E(x)$  is called a base of neighborhoods of  $x$  (or a fundamental system of neighborhoods of  $x$ ) if and only if every neighborhood in  $E(x)$  contains one in  $\beta(x)$

A non empty class of subsets  $\beta_x(x)$  containing the point  $x$  in a set  $X$  is a base of neighborhoods of  $x$  for a certain topology if and only if the intersection of two members in  $\beta_x(x)$  contains an element in  $\beta_x(x)$ .

In a linear space  $X$  a subset  $U$  is said to be

1) absorbent if and only if

for every  $x \in X$  there exist  $\rho > 0$  such that  $\rho^{-1}x \in U$ .

2) circled (balanced) if and only if

$\rho x \in U$  for every  $x \in U$  and every  $|\rho| \leq 1$ .

3) convex if and only if

$\lambda x + \beta y \in U$  whenever  $x, y \in U$  and  $\lambda + \beta = 1$ ,  $\lambda, \beta \geq 0$ .

4) absolutely convex if and only if

$\lambda x + \beta y \in U$  whenever  $x, y \in U$  and  $|\lambda| + |\beta| \leq 1$ .

The circled hull of  $U$  is defined by

$\{\lambda x, x \in U \text{ and } |\lambda| \leq 1\}$ .

The convex hull  $C(U)$  of  $U$  is the set of all elements of the form

$$\sum_{i=1}^n \lambda_i x_i \quad \text{with } \lambda_i \geq 0, \quad \sum_{i=1}^n \lambda_i = 1, \quad x_i \in U.$$

The absolutely convex hull  $\Gamma(U)$  of  $U$  consisting of all elements of the form

$$\sum_{i=1}^n \lambda_i x_i \quad \text{with } \lambda_i \in K, \quad \sum_{i=1}^n |\lambda_i| \leq 1, \quad x_i \in U.$$

Remark:

The set is absolutely convex if and only if it is convex and balanced.

A semi norm  $P$  on the linear space  $X$  is a mapping satisfying the norm axioms except that

$$P(x) = 0 \quad \text{need not imply that } x = 0.$$

Every balanced, absorbent, and convex set  $U$  defines a semi norm  $p_U$  called Minkowski functional of  $U$  defined by,

$$p_U(x) := \inf_{\lambda > 0} \{ \lambda, \lambda^{-1}x \in U \}$$

for which

$$U = \{x, p_U(x) < 1\}.$$

On the other hand starting from a semi norm  $p(x)$  on  $X$  the sets

$$U_{p,\varepsilon} = \{x, p(x) < \varepsilon\}$$

are absolutely convex and absorbent.