Ain Shams University

ON TENSOR PRODUCTS OF BANACH SPACES AND SOME APPLICATIONS

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INTRODUCTION

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Grothendiek's book" Produits tensoriels topologiques et espaces nucléaires" published in 1956 is one of the very few books which have deeply influenced the course of Functional Analysis. It not only demonstrated the enormous possibilities for the using of tensor products in Banach space theory, but it anticipated the study of Banach spaces in terms of finite dimensional subspaces.

Apparently few researchers noted its importance. This, in spite of the fact that Grothendieck's theory of locally convex tensor products and nuclear spaces had already won great appreciation.

Tensor products apparently appeared in Functional Analysis for the first time during the late thirties in the work of Murray and John von Neumann on Hilbert spaces. The first systematic study of classes of norms on tensor products of Banach spaces is due to R. Schatten who in 1943 continued his work in a series of papers.

The projective tensor norm Π and injective tensor norm ε proved their usefulness rather quickly through simple examples involving vector valued function spaces. New results about general tensor norms were obtained around 1970.

Back in 1965 A. Pietsch has raised the following questions.

1) Must a locally convex space X be nuclear if the two spaces $X \otimes_{\varepsilon} X$ and $X \otimes_{\pi} X$ are homeomorphically identical?.

2) What properties a locally convex space Y must posses in order that the projective tensor product $X\otimes_\pi Y$ and the injective tensor product $X \otimes_\varepsilon Y$ are homeomorphically identical?.

3) Is a locally convex space X nuclear, if all bilinear forms on $X \times X$ are nuclear?.

The answer of the first question was included in Pisier's work in Banach space theory (starting around 1975) which finally drew new attention to the tensor product approach. A highlight is certainly his solution of the problem of finding an infinite dimensional Banach spaces X for which

$$X\otimes_\varepsilon X \cong X\otimes_\pi X$$

holds isomorphically.

H. Jarchow and K. John have solved the second and the third questions in 1988 and 1991 see [5, 6].

The thesis consists of six chapters

Chapter 0:

It contains some basic definitions and construction of some locally convex topologies that will be used.

Chapter 1:

Our main interests of study are the properties of the following two topologies on the tensor product $X \otimes Y$ namely,

1- The strongest locally convex topology on the tensor product $X\otimes Y$ such that the canonical bilinear mapping ϕ from $X\times Y$ into $X\otimes Y$ defined by $\phi(x, y) = x\otimes y$ is continuous. This topology is known as the projective tensor topology π and led us to the study of the concept of nuclear operators from X into Y.

2- The coarsest locally convex topology on the tensor product $X\otimes Y$ such that the canonical bilinear mapping ϕ (from $X\otimes Y$ into the space of all continuous bilinear forms $\beta_e(X^*_s \times Y^*_s)$) defined by $[\phi(\sum_{i=1}^n x_i \otimes y_i)](f, g) = \sum_{i=1}^n f(x_i)\, g(y_i)$ is continuous. This topology is known as the injective tensor product ε and led us to the study of the concept of approximable operators which is a subclass of the class of compact operators from X into Y.

Chapter 2:

In this chapter we study some questions which raised by Banach such as: Is every compact operator approximable?. Is the canonical imbedding of $X^*\hat{\otimes}_\pi Y$ onto the space of all nuclear operators $N(X; Y)$ is injective? Is the trace defined for nuclear operators?. These questions find an answer with the help of Grothendieck's approximation property.

Chapter 3:

In this chapter we study a partial ordering mentioned in [21] among finite collections of elements of a Banach space X as follows:

$$y = \{y_i\}_{i=1}^n \le x = \{x_i\}_{i=1}^m$$

if and only if for every $f\in X^*$,

$$\left(\sum_{i=1}^n |f(y_i)|^2 \right)^{\frac{1}{2}} \le \left(\sum_{i=1}^m \ |f(x_i)|^2 \right)^{\frac{1}{2}} .$$

He showed that this happens if and only if y is the image of x under an n×m matrix.

In terms of this partial ordering G. Pisier had given a necessary and sufficient condition for an operator T to be factored through a Hilbert space. During this chapter we also study the 3^{rd} topology which is lying between the projective and injective topologies induced by the norm $\| \circ \|_*$ (see [21]), has a great connection with operators factoring through Hilbert spaces.

Chapter 4:

In this chapter we study the relations between the injective and projective tensor products, nuclear mappings between locally convex spaces and the concept of nuclear spaces.

Chapter 5:

In this chapter we use infinite matrices defining bounded linear operators on the space c_o of convergent to zero sequences to define operators on the classical Banach space $c_o(X)$ and define a partial ordering among convergent to zero sequences according to values depending on functionals and prove that this partial ordering could be obtained by application of infinite matrices.

We get an analogous result to that mentioned in [21].

For the completeness of our work we had to rewrite many proofs in a way compatible to the sequel of the thesis.

CHAPTER (0)

SOME NOTATIONS AND BASIC DEFINITIONS

CHAPTER 0

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0.1 Locally Convex Spaces.

0.1.1 Bases of Neighborhoods

If $E(x)$ is the class of all neighborhoods of a point x of the topological space X, a subclass $\beta(x)$ of $E(x)$ is called a base of neighborhoods of x (or a fundamental system of neighborhoods of x) if and only if every neighborhood in $E(x)$ contains one in $\beta(x)$

A non empty class of subsets $\beta_x(x)$ containing the point x in a set X is a base of neighborhoods of x for a certain topology if and only if the intersection of two members in $\beta_x(x)$ contains an element in $\beta_x(x)$.

In a linear space X a subset U is said to be

1) absorbent if and only if

$$\text{for every } x \in X \text{ there exist } \rho > 0 \text{ such that } \rho^{-1}x \in U.$$

2) circled (balanced) if and only if

$$\rho x \in U \qquad \text{for every } x \in U \text{ and every } |\rho| \le 1 \quad .$$

3) convex if and only if

$$\lambda x + \beta y \in U \qquad \text{whenever } x,\ y \in U \text{ and } \lambda + \beta = 1,\ \lambda,\ \beta \ge 0.$$

4) absolutely convex if and only if

$$\lambda x + \beta y \in U \qquad \text{whenever } x,\ y \in U \text{ and } |\lambda| + |\beta| \le 1 \quad .$$

The circled hull of U is defined by

$$\{\lambda x,\ x \in U \text{ and } |\lambda| \le 1\}.$$

The convex hull C(U) of U is the set of all elements of the form

$$\sum_{i=1}^{n} \lambda_i x_i \qquad \text{with } \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ x_i \in U \qquad .$$

The absolutely convex hull Γ(U) of U consisting of all elements of the form

$$\sum_{i=1}^{n} \lambda_i x_i \qquad \text{with } \lambda_i \in K, \ \sum_{i=1}^{n} |\lambda_i| \leq 1, \ x_i \in U \qquad .$$

Remark:

The set is absolutely convex if and only if it is convex and balanced.

A semi norm P on the linear space X is a mapping satisfing the norm axioms except that

$$P(x) = 0 \qquad \text{need not implies that } x = 0.$$

Every balanced, absorbent, and convex set U defines a semi norm p_U called Minkowski functional of U defined by,

$$p_U(x) := \inf_{\lambda > 0} \{\lambda, \ \lambda^{-1} x \in U\}$$

for which

$$U = \{x, \ p_U(x) < 1\}.$$

On the other hand starting from a semi norm p(x) on X the sets

$$U_{p,\varepsilon} = \{x, \ p(x) < \varepsilon\}$$

are absolutely convex and absorbent.