GRAVITATIONAL WAVES
IN GENERAL RELATIVITY THEORY

THESIS

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By

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Gravitational Waves

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M.Sc. COURSES

STUDIED BY THE AUTHOR (1971 - 1972)

(i) Relativistic Cosmology — 2 hours per wee
(during the whole academic year)

(ii) Space dynamics — 2 hours per wee

(iii) General conceptions of continuous mechanics of continuous medium — 2 hours per wee

(iv) Theory of functions of matrices — 2 hours per wee

(v) Functional analysis — 2 hours per wee

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Chapter I

I N T R O D U C T I O N

One of the most important problems in the study of general relativity is the theory of gravitational waves. The theory of gravitational waves was initiated by Einstein himself (1917-1919) in two papers on the approximate wave like solutions of his empty space field equations.

The investigation of gravitational radiation in general relativity theory is hampred by the lack of an invariant radiation - for the non-linearty of the gravitational field equations of general relativity - which must be distinguishable, mathematically from a peculiar choice of the coordinate system, and physically, from a peculiar motion of the observer. In a covariant, non-linear, theory the definition of gravitational radiation should not depend on the weakness of the fields or on special coordinate conditions. In this thesis an invariant definition will be proposed in chapter III.

In the study of gravitational fields, using general relativity, the essential information about the field is mainly given by the Riemann- Christoffel tensor R_{abcd}. Hence, in order to get an appropriate basis for the investigation of gravitational radiation, one must look clearly into the algebraic and differential properties of this tensor. Such an investigation depends largely on the exploitation of analogies in the gravitational field to the electromagnetic field. This requires the transformation of many familiar results of electromagnetic theory into some other unfamiliar forms before these analogies can be worked out. For instance, some often-used techniques of electromagnetic theory. Such as the resolution into Fourier components, are not natural for the (non-linear) gravitational fields. The role of the Riemann tensor in this study is explained in chapter II.

In chapter IV plane gravitational waves are defined and a general plane-wave metric is obtained and the properties of plane-wave space-times are studied in detail.

In particular, their characterization as "plane" is justified further by the construction of "sandwich waves" bounded on both sides by (null) hypersurfaces in flat space-time.

In order to get some insight into the behaviour of gravitational fields we have studied the linearized theory of gravitation (chapter V). The first calculations along these lines, for the linearized gravitational field were carried out by Sachs (1961) using the super potential method developed earlier by Sachs and Bergmann (1958) Si-milar calculation for the electromagnetic field has been given independently by Lyuboshity and Smorodinskii (1962). A mush more general formalism for the linearized theory has been developed by Pirani (1964).

Notation Used:

1- The absolute derivatives of vector and tensor fields taken along the curve

$$x^a = x^a(u),$$

u beeing a parameter, are defined in the following manner:

$$\frac{\delta A^a}{\delta u} = \frac{dA^a}{du} + \Gamma^a_{bc} A^b \frac{dx^c}{du} \qquad (1\text{-}1)$$

$$\frac{\delta A_a}{\delta u} = \frac{dA_a}{du} - \Gamma^b_{ac} A_b \frac{dx^c}{du} . \qquad (1\text{-}2)$$

$$\frac{\delta B^{ab}}{\delta u} = \frac{dB^{ab}}{du} + \Gamma^a_{cd} B^{cb} \frac{dx^d}{du} + \Gamma^b_{cd} B^{ac} \frac{dx^d}{du} \qquad (1\text{-}3)$$

$$\frac{\delta B^a_b}{\delta u} = \frac{dB^a_b}{du} + \Gamma^a_{cd} B^c_b \frac{dx^d}{du} - \Gamma^c_{bd} B^a_c \frac{dx^d}{du} \qquad (1\text{-}4)$$

$$\frac{\delta B_{ab}}{\delta u} = \frac{B_{ab}}{du} - \Gamma^c_{ad} B_{cb} \frac{dx^d}{du} - \Gamma^c_{bd} B_{ac} \frac{dx^d}{du} \qquad (1\text{-}5)$$

2- Partial and covariant derivatives are denoted by

$$\partial_a A^{\cdots} = A^{\cdots}{}_{,a} \quad \text{and} \quad \cdots \qquad (1\text{-}6)$$

$$\nabla_a A^{\cdots} = A^{\cdots}{}_{;a} \qquad (1\text{-}7)$$

3- Symmetrization is denoted by round brackets

$$A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba}) \qquad (1\text{-}8)$$

4- Antisymmetrization is denoted by square brackets

$$A_{[ab]} = \frac{1}{2} (A_{ab} - A_{ba}) \text{ , for tensors of the second order} \qquad (1\text{-}9)$$

$$A_{[abc]} = \frac{1}{3} (A_{abc} + A_{bca} + A_{cab} - A_{cba} - A_{acb} - A_{bac}) \qquad (1\text{-}10)$$

for tensors of the third order.

5- The metric tensor is denoted by g_{ab} and its determinant is denoted by g , the Minkowski's metric tensor is denoted by η :

$$\eta_{ab} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \qquad (1\text{-}11)$$

The Kronecker delta is given by:

$$\delta^{b}_{a} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \qquad (1\text{-}12)$$

6- The sign of the Riemann tensor is specified by the Ricci identity

$$\nabla_{[c} \nabla_{d]} A_{b} = \frac{1}{2} R^{a}_{\ bcd} A_{a} \qquad (1\text{-}13)$$

The Ricci tensor is given by:

$$R_{bd} = R^{a}_{\ bad} \qquad (1\text{-}14)$$

and the curvature scalar is given by:

$$R = R^{b}_{b} \qquad (1\text{-}15)$$

The Einstein tensor is denoted by:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \qquad (1-16)$$

The Weyl tensor C_{abcd}, is defined by

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2 \delta^{[a}{}_{[c} \left(R^{b]}{}_{d]} - \frac{1}{6} R \delta^{b]}{}_{d]} \right) \qquad (1-17)$$

It has the property that

$$C^{ad}{}_{ad} = 0$$

The atternating (oriented) tensor is

$$\epsilon_{abcd} = \epsilon_{[abcd]} \qquad (1-18)$$

with $\epsilon_{1234} = (-g)^{\frac{1}{2}} > 0$

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Chapter II

MEANING OF GRAVITATIONAL WAVES

It is now belived that, according to Einstein's general theory of relativity, when a particle is accelerated or a body is deformed in shape, gravitational waves are emitted. These waves are assumed to travel through empty space with the speed of light, carrying with them the information that a change has occured in the gravitational field of the source.

Before entering into technical discussions, it is desirable to clarify the meaning of the term <u>gravitational waves</u>, not by a formal definition, but by some physical considerations.

For instance, if a man, standing on the earth, holds in his hand a heavy club. At first, the club ~~h~~hangs down towards the ground, but at a certain moment the man raises it quickly over his head. The club itself would produce a gravitational field, and the action of the man changes this field, not only in his neighbourhood but through the whole universe. This change in the gravitational field of the moving club travels out into space with the speed of light. This moving disturbance resembles what is known as a "gravitational wave".

1- Analogy with electromagnetic waves

As in electromagnetic theory, the idea of electromagnetic