



**Ain Shams University
Faculty of Education
Mathematics Department**

Properties Of Derivations On KU-Algebra

A Thesis Submitted to Mathematics Department, Faculty of Education,
Ain Shams University in Partial Fulfillment of the Requirement for the
Master's Degree in Teacher Preparation in Science
(Pure Mathematics)

By

Ahmed abd eldayem ali elgamal

Supervised by

**Dr. Samy M . Mostafa
Ass. Prof. of pure Mathematics.
Faculty of education
Ain Shams University**

**Dr. Ragab Abd – Al kader omar
Lecture . of pure Mathematics.
Faculty of education
Ain Shams University**

(2016)



Ain Shams University
Faculty of Education
Mathematics Department

Candidate: Ahmed abd eldayem ali elgamal.

Thesis Title: Properties Of Derivations On Ku-Algebra

Degree: Master's Degree in Teacher Preparation in Science.
(Pure Mathematics)

Supervisors:

Signature

Dr. Samy M . Mostafa
Ass. Prof. of pure Mathematics.
Faculty of education
Ain Shams University

Dr. Ragab Abd – Al kader omar
Lecture . of pure Mathematics.
Faculty of education
Ain Shams University

(2016)

Acknowledgements

I am deeply thankful to God ,by the grace of whom every success in my life was possible.

My gratitude and deep thanks to **Dr.samy M.Mostafa**, associate professor, at the Department of Mathematics, faculty of Education ,Ain shams University,for his kind supervision and for his invaluable help during the preparation of this thesis . I wish to express my deepest gratitude to him ,for suggesting the topic of the thesis ,for his patience,criticism,advice and support during the preparation of this thesis.

My gratitude and deep thanks to **Dr. Ragab Abd – Al kader omar** lecture of Mathematics, faculty of Education ,Ain shams University for his interest and fruitful help throughout this work.

My gratitude and deep thanks to **prof.Dr.Aosman mohammed**.the head of Math department ,faculty of Education ,Ain shams University for his interest and fruitful help throughout this work.

My gratitude and deep thanks to my family and every one who helped me through this work.

Contents

Summary	I
Chapter one: Basic concepts and results	
1.1 Introduction to BCK – algebras	1
1.2 Bounded BCK-algebras.....	3
1.3 BCK-ideals	5
1.4 Homomorphism and isomorphism on BCK / BCI –algebras	6
1.5 KU-algebras and some related properties.....	8
1.6 Some fuzzy concepts.....	13
1.7 Fuzzy KU- ideals of KU-algebras.....	15
Chapter two: Survey on some types of algebraic structures with derivation	
2.1 Left Derivation of BCI-algebra.....	19
2.2. ON f -derivations of BCI-algebras.....	26
2.3 Derivation of BCC- algebra	30
2.4 A Note on f -Derivations of BCC-Algebr.....	35
2.5 Derivations on d-Algebras	37
2.6 Derivations of B-algebra.....	40
2.7 Derivations of KUS-algebras	43
2.8 Derivations on TM-algebras	50
2.9 Derivations of incline algebras.....	61
2.10 Derivations of boolean lgebras.....	66
Chapter three: Properties of Derivations on KU-Algebras	
3.1 Some properties of KU-algebras	75
3.2 The derivations on KU-algebra.....	76
3.3 t - Derivation ON KU-algebra.....	81
Chapter four : Fuzzy derivations KU-ideals on KU-Algebras	
4.1 Fuzzy derivations KU- ideals of KU-algebras.....	87
4.2 Characterization of fuzzy left derivations KU - ideal by their level KU – ideals.....	92
4.3 Cartesian product of fuzzy left derivations KU-ideals.....	94
Chapter five : Q-Fuzzy derivations KU-ideals on KU-algebras	
5.1 Q -Fuzzy derivations KU- ideals of KU-algebras.....	97
5.2 Image (Pre-image) of a Q - fuzzy derivations KU-ideals under homomorphism.....	101
5.3 Cartesian product of a Q - fuzzy derivations KU-ideals	104
Reference	108
Appendix	112

PREFACE

As it is well known, BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [21, 22, 26] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. The essential difference between BCK-algebras and BCI-algebras lies in the following: The Element 0 is the least element in BCK-algebras, while it is a minimal element in BCI-algebras. The class of all BCK-algebras is a quasivariety. Is'eki posed an interesting problem (solved by Wro'nski [60]) whether the class of BCK-algebras is a variety. In connection with this problem, Komori [39] introduced a notion of BCC-algebras, and Dudek [15] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [16] introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences.

Prabpayak and Leerawat [55, 56] introduced a new algebraic structure which is called KU-algebras. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties. Zadeh [63] introduced the notion of fuzzy sets. the study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy groups by Rosenfeld [57]. Xi [61] introduced the notion of fuzzy Bck-algebra. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory and topology.

Mostafa et al [46] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Akram et al and Yaqoob et al [2, 62] introduced the notion of cubic sub-algebras and ideals in KU-algebras. They discussed relationship between a cubic subalgebra and a cubic KU-ideal.

Several authors [7, 8, 10, 11, 20, 37] have studied derivations in rings and near rings. Jun and Xin [31] applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some of its properties, defined a d -derivation ideal and gave conditions for an ideal to be d -derivation.

Later, Hamza and Al-Shehri [1], defined a left derivation in *BCI*-algebras and investigated a regular left derivation. Zhan and Liu [65] studied *f*-derivations in *BCI*-algebras and proved some results.

G. Muhiuddin etl [48 , 49] introduced the notion of (α, β) -derivation in a *BCI*-algebra and investigated related properties. They provided a condition for a (α, β) -derivation to be regular. They also introduced the concepts of a $d_{(\alpha, \beta)}$ -invariant (α, β) -derivation and α -ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α, β) -derivations. Moreover, they studied the notion of *t*-derivations on *BCI*-algebras and obtained some of its related properties. Further, they characterized the notion of *p*-semisimple *BCI*-algebra *X* by using the notion of *t*-derivation.

The aim in this thesis is to introduce the (ℓ, r) $((r, \ell))$ -derivation , *t*-derivation in *KU*-algebra, fuzzy (left and right) derivations of *KU*-ideals in *KU*-algebras , homomorphic image (preimage) of fuzzy left (right)-derivations *KU*-ideals in *KU*-algebras under homomorphism of a *KU* –algebras , the cartesian product of fuzzy left (right) derivations *KU* - ideals in cartesian product of *KU* – algebras and their generalizing results .

This thesis is broadly divided into five chapters .

Chapter one highlights some basic definitions and results available in the standard literature.

Chapter two is a survey on some types of algebraic structures with derivation, which collect all the necessary preliminaries which will be useful in our discussions in the main part of the thesis.

The main part of the thesis are given in chapters 3 , 4 and 5 .

In chapter three, the notion of (ℓ, r) $((r, \ell))$ -derivations and *t*-derivation of a *KU*-algebra are introduced, and some related properties are investigated. Also, we consider

regular derivations and the d-invariant on ideals of KU-algebras .We also characterized kernel d by derivations.

In Chapter four we give the concepts of fuzzy (left and right) derivations of KU-ideals in KU-algebras and homomorphic image (preimage) of fuzzy left (right)-derivations KU-ideals in KU-algebras under homomorphism of a KU –algebras are introduced. Also the homomorphic images and inverse images of fuzzy (left and right) derivations KU-ideals in KU – algebras are discussed. Furthermore, the concept of the Cartesian product of fuzzy left (right) derivations KU - ideals in cartesian product of KU – algebras are given. Many related results have been derived.

In Chapter five we introduce the notion of Q - fuzzy KU-ideals of KU-algebras and several results are presented in this regard. The image, pre-image, and cartesian product of Q - fuzzy KU-ideals of KU-algebras are defined.

LIST OF PUBLICATIONS

1-S .M. Mostafa, R. Omar, A.abd -eldayem

Properties of derivations on KU-algebras ,

JOURNAL OF ADVANCES IN MATHEMATICS Vol .10, No

February 5 ,2015(3085-3097)

2- Samy M. Mostafa ; A.abd-eldayem

Fuzzy derivations KU-ideals on KU-algebras

eprint arXiv:1506.04475 ,Published in Jun 2015 (1-21)

3- Samy M. Mostafa ; A.abd-eldayem

Q-fuzzy derivations KU-ideals on KU-algebras

Algebra Letters,septemper1,2015(1-18)

Chapter 1

Basic concepts and results

In this chapter we collect all the necessary preliminaries which will be useful in our discussions in the main part of the thesis.

§ (1.1) Introduction to BCK – algebras

In this section, we review some definitions and results that are needed without proofs.

These results in this section are taken from [23,24,25,26,31,32,42,56,58]

Preliminaries

Definition 1.1.1[24] : Let X be a set with a binary operation “ $*$ ” and a constant 0 , then $(X, *, 0)$ is called a BCI -algebra ,if it satisfies the following axioms:

$$(BCI -1) ((x * y) * (x * z)) * (z * y) = 0$$

$$(BCI -2) (x * (x * y)) * y = 0$$

$$(BCI -3) x * x = 0$$

$$(BCI -4) x * y = 0 \text{ and } y * x = 0 \text{ implies } x = y$$

for all $x, y, z \in X$

If a BCI -algebra X satisfies the identity $0 * x = 0$, for all $x \in X$, then X is called a BCK algebra. It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. For brevity X is called a BCK - algebra .

A binary relation $\leq$ in X is defined by : $x \leq y$ if and only if $x * y = 0$, then by using this relation we can write the definition of BCK algebra as follows :

$(X, *, 0)$ is a BCK - algebra if and only if it satisfies that :

$$(BCI \text{ `}_1) : ((x * y) * (x * z)) \leq z * y$$

$$(BCI \text{ `}_2) : (x * (x * y)) \leq y$$

$$(BCI_3) : x \leq x,$$

$$(BCI_4) : x \leq y \text{ and } y \leq x \text{ implies } x = y$$

$$(BCI_5) : 0 \leq x.$$

In a BCK - algebra $(X, *, 0)$, the following properties are satisfied :

1. $x \leq y$ implies $x * z \leq y * z$
2. $x \leq y$ and $y \leq z$ imply $x \leq z$
3. $(x * y) * z = (x * z) * y$
4. $(x * y) \leq z$ implies $x * z \leq y$
5. $(x * z) * (y * z) \leq x * y$.
6. $x \leq y$ implies $x * z \leq y * z$.
7. $(x * (x * y)) * (y * x) \leq x * (x * (y * (y * x)))$
8. $x * y \leq x$
9. $x * 0 = x$ for all $x, y, z \in X$.

Example 1.1.2 : Let $X = \{0,1,2,3,4\}$ in which $*$ is defined by the following table :

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	1	1	1	0

Then X is BCK - algebra .

Theorem 1.1.3 [24] : An algebra $(X, *, 0)$ of type $(2, 0)$ is a BCK - algebra if and only if it satisfies the following conditions :

$$(BCI_1) : ((x * y) * (x * z)) * (z * y) = 0 \text{ and } x * (0 * y) = x$$

(BCI₄) : $x * y = 0 = y * x$ implies $x = y$, for all $x, y, z \in X$.

For any $x, y \in X$ denote $x \wedge y = y * (y * x)$

Obviously $x \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$. But in general, $x \wedge y \neq y \wedge x$

Definition 1.1.4 [24] : Let $(X, *, 0)$ be a BCK – algebra, and let S be a non – empty subset of X , then S is called a sub - algebra of X , if for all $x, y \in S$, $x * y \in S$, i.e S is closed under the binary operation $*$ of X .

Example 1.1.5 : Let $X = \{0,1,2\}$ in which $*$ is defined by the following table :

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $S = \{0,2\}$ is BCK sub-algebra of X .

Theorem 1.1.6 [24] : Suppose that $(X, *, 0)$ is a BCK - algebra and let S be a sub - algebra of X :
then :

- (i) $0 \in S$
- (ii) $(S, *, 0)$ is also a BCK – algebra
- (iii) X is a subalgebra of X
- (iv) $\{0\}$ is also a subalgebra of X .

§ 1.2 Bounded BCK-algebras

The main results in this section are taken from [23]

Definition 1.2.1 : If there is an element 1 of a BCK – algebra X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is called unit of X . A BCK – algebra with unit is said to be bounded.

Example 1.2.2 : Let $X = \{0,1,2,3,4\}$ in which $*$ is defined by the following table :

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X, *, 0)$ is a bounded BCK-algebra with unit 4 .

Note : In a bounded BCK – algebra X , we denote $1 * x$ by N_x .

Definition 1.2.3 : For a bounded BCK – algebra X , if an element x satisfies $NN_x = x$, then x is called an involution .

Theorem 1.2.4 In a bounded BCK – algebra X , we have :

- (a) $N_1 = 0$, $N_0 = 1$,
- (b) $NN_x \leq x$
- (c) $N_y * N_x \leq y * x$
- (d) $y \leq x$ implies $N_x \leq N_y$
- (e) $N_{x*y} = N_{y*x}$
- (f) $NNN_x = N_x$

Theorem 1.2.5 : In a bounded BCK – algebra X , we have $x * N_y = y * N_x$,

for all x and $y \in S(X)$ where $S(X)$ is the set of all involutions of a bounded BCK- algebra .

Note that : $NN_0 = N_1 = 0$, and $NN_1 = N_0 = 1$, then the elements 0 and 1 are contained in $S(X)$. Hence $S(X)$ is non - empty .

Theorem 1.2.6 : For any bounded BCK - algebra X , we have $S(X)$ is a bounded sub-algebra of X .

Theorem 1.2.7: A BCI -algebra X satisfying $x * (y * z) = (x * y) * z$ is a group in which every element is an involution .

§ 1.3 BCK-ideals

The main results in this section are taken from [27]

Definition 1.3.1 : A non - empty subset I of a BCK-algebra X is called an BCK ideal of X , if it satisfies the following conditions :

- (1) $0 \in I$
- (2) $x * y \in I$, $y \in I$ implies $x \in I$, for all $x, y \in X$.

Example 1.3.2: Let $X = \{0,1,2\}$ in which $*$ is defined by the following table :

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is BCK-algebra , and $\{0\}$, X , $\{0,1\}$, $\{0,2\}$ are all ideals of X .

Example 1.3.3: Let $X = \{0,1,2,3\}$ in which $*$ is defined by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Then $(X, *, 0)$ is BCK-algebra , it has only two ideals $\{0\}$ and X .

The following results gives some properties of ideals In BCK algebra .

Theorem 1.3.4: Suppose I is an ideal of a BCK-algebra X and $x \in I$, if $y \leq x$ then $y \in I$.

Definition 1.3.5 Given a BK-algebra $(X, *, 0)$ and given elements a, b of X , we define

$$A(a, b) = \{x \in X : x * a \leq b\}.$$

Theorem 1.3.6: Suppose I is a non-empty subset of a BCK-algebra X , then I is an ideal of X if and only if, for any $x, y \in I$, $A(x, y) \subseteq I$.

Corollary 1.3.7: I is an ideal of X if and only if for all $x, y \in I$, $(z * x) * y = 0$ implies $z \in I$

§ 1.4. Homomorphisms and isomorphisms on BCK/BCI-algebras

The main results in this section are taken from K.Iseki [25], Y.B.Jun [30] and [41].

Definition 1.4.1: Let $(X, *, 0)$ and $(X', *, 0)$ are two BCK-algebras. A mapping $f : X \rightarrow X'$ is called a homomorphism from X into X' if, $\forall x, y \in X, f(x * y) = f(x) *' f(y)$. If f is onto i.e. $f(X) = X'$, then f is called an epimorphism. If f is both an epimorphism and one to one, then f is called an isomorphism.

Denote the set of all homomorphisms from X to X' by $\text{Hom}(X, X')$, this set is always non empty since it contains the zero hom. $0 : X \rightarrow X'$, which is defined by $0 * x = x' \forall x \in X$. Let $f \in \text{Hom}(X, X')$ and $(B \subseteq X')$ The set $\{x \in X : f(x) \in B\}$ is denoted by $f^{-1}(B)$ and is called the inverse image of B under f . In particular, $f^{-1}(\{0'\}) = \{x \in X : f(x) = 0'\}$ is called the kernel of f . In case $X = X'$, a homomorphism is called an endomorphism and isomorphism is referred as an automorphism. The identity mapping $1 : X \rightarrow X'$ is clearly an endomorphism.

Theorem 1.4.2: Suppose $f : X \rightarrow X'$ is an homomorphism, then :

- (a) $f(0) = 0'$,
- (b) f is an isotone, i.e. if $x \leq y \Rightarrow f(x) \leq f(y)$

Theorem 1.4.3 : Let $(X, *, 0)$ and $(X', *, 0)$ be two BCK-algebras and let B be an ideal of X' then for any $f \in \text{Hom}(X, X')$, $f^{-1}(B)$ is an ideal of X .

Corollary 1.4.4 : $\text{Ker}(f)$ is an ideal of X .

Theorem 1.4.5 : Suppose I be a proper ideal of a BCK-algebra X , then for any BCK-algebra X' there exists $f \in \text{Hom}(X, X')$ such that $\text{Ker}(f) = I$ if and only if I is obstinate.

Theorem 1.4.6 : Suppose X, Y and Z are BCK-algebras, let $h: X \rightarrow Y$ be an epimorphism and $g \in \text{Hom}(X, Z)$, if $\text{Ker}(h) \subseteq \text{Ker}(g)$ then there exists a unique homomorphism $f: X \rightarrow Z$ such that $f \circ h = g$

Definition 1.4.7 : Given two BCK-algebras X and X' . If there exists an epimorphism $f: X \rightarrow X'$ then we call X to be homomorphic to X' , written $X \sim X'$. If there exists an isomorphism $f: X \rightarrow X'$ then we call X to be isomorphic to X' , written $X \cong X'$.

Theorem 1.4.8: Given a BCK-algebras X . If I is an ideal of X , then the quotient algebra X/I is a homomorphic image of X .

Theorem 1.4.9: Given two BCK-algebras X and Y . If $f: X \rightarrow Y$ is an epimorphism, then $X / \text{Ker}(f) \cong Y$.

Definition 1.4.10 A mapping f of a BCI-algebra X into itself is called an endomorphism of X if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(0) = 0$. Especially, f is monic if for any $x, y \in X$, $f(x) = f(y)$ implies that $x = y$.

For a BCI-algebra X , denote by X_+ (resp., $G(X)$) the BCK-part (resp., the BCI-G part) of X , that is, $X_+ = \{x \in X / 0 \leq x\}$ (resp., $G(X) = \{x \in X / 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$. If $X_+ = \{0\}$, then X is called a p -semisimple BCI-algebra.

Theorem 1.4.11 In a p -semisimple BCI-algebra X , the following hold:

- (1) $(x * z) * (y * z) = x * y$
- (2) $0 * (0 * x) = x$
- (3) $x * (0 * y) = y * (0 * x)$
- (4) $x * y = 0$ implies $x = y$
- (5) $x * a = x * b$ implies $a = b$
- (6) $a * x = b * x$ implies $a = b$