Ain Shams University
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Properties Of Derivations On KU-Algebra

A Thesis Submitted to Mathematics Department, Faculty of Education, Ain Shams University in Partial Fulfillment of the Requirement for the Master's Degree in Teacher Preparation in Science
(Pure Mathematics)

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Contents

PREFACE

As it is well known, BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [21, 22,26] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. The essential difference between BCK-algebras and BCI-algebras lies in the following: The Element 0 is the least element in BCK-algebras, while it is a minimal element in BCIalgebras .The class of all *BCK*-algebras is a quasivariety. Is´*e*ki posed an interesting problem (solved by Wro´*n*ski [60]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Komori [39] introduced a notion of *BCC*-algebras, and Dudek [15] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [16] introduced a new notion of ideals in *BCC*-algebras and described connections between such ideals and congruences .

Prabpayak and Leerawat [55,56] introduced a new algebraic structure which is called KU-algebras. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties. Zadeh [63] introduced the notion of fuzzy sets.the study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy groups by Rosenfeld [57].Xi [61] introduced the notion of fuzzy Bck-algebra. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory and topology.

Mostafa et al [46] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Akram et al and Yaqoob et al [2, 62] introduced the notion of cubic sub-algebras and ideals in KU-algebras. They discussed relationship between a cubic subalgebra and a cubic KU-ideal.

Several authors [7,8,10 ,11, 20 , 37] have studied derivations in rings and near rings. Jun and Xin [31] applied the notion of derivations in ring and near-ring theory to *BCI*-algebras, and they also introduced a new concept called a regular derivation in *BCI*-algebras. They investigated some of its properties, defined a d -derivation ideal and gave conditions for an ideal to be d-derivation.

Later, Hamza and Al-Shehri [1], defined a left derivation in *BCI*-algebras and investigated a regular left derivation. Zhan and Liu [65] studied f-derivations in BCI-algebras and proved some results.

G. Muhiuddin etl [48 , 49] introduced the notion of (α,β)-derivation in a BCI-algebra and investigated related properties. They provided a condition for a (α,β)-derivation to be regular. They also introduced the concepts of a $d_{(\alpha,\beta)}$- invariant (α,β) -derivation and α-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α,β) - derivations. Moreover, they studied the notion of *t*-derivations on BCI-algebras and obtained some of its related properties. Further, they characterized the notion of *p*-semisimple BCI-algebra *X* by using the notion of *t*-derivation.

The aim in this thesis is to introduce the (ℓ,r) $((r,\ell))$ -derivation , t-derivation in KU-algebra, fuzzy (left and right) derivations of KU-ideals in KU–algebras , homomorphic image (preimage) of fuzzy left (right)-derivations KU-ideals in KU-algebras under homomorprhism of a KU –algebras , the cartesian product of fuzzy left (right) derivations KU - ideals in cartesian product of KU – algebras and their generalizing results .

This thesis is broadly divided into five chapters .

Chapter one highlights some basic definitions and results available in the standard literature.

Chapter two is a survey on some types of algebraic structures with derivation, which collect all the necessary preliminaries which will be useful in our discussions in the main part of the thesis.
The main part of the thesis are given in chapters 3 , 4 and 5 .

In chapter three, the notion of (ℓ,r) $((r,\ell))$ -derivations and t-derivation of a KU-algebra are introduced, and some related properties are investigated. Also, we consider regular derivations and the d-invariant on ideals of KU-algebras .We also characterized kernel d by derivations.

In Chapter four we give the concepts of fuzzy (left and right) derivations of KU-ideals in KU–algebras and homomorphic image (preimage) of fuzzy left (right)-derivations KU-ideals in KU-algebras under homomorprhism of a KU –algebras are introduced. Also the homomorphic images and inverse images of fuzzy (left and right) derivations KU-ideals in KU – algebras are discussed. Furthermore, the concept of the Cartesian product of fuzzy left (right) derivations KU - ideals in cartesian product of KU – algebras are given. Many related results have been derived.

In Chapter five we introduce the notion of *Q*- fuzzy KU-ideals of KU-algebras and several results are presented in this regard. The image, pre-image, and cartesian product of *Q*- fuzzy KU-ideals of KU-algebras are defined.

LIST OF PUBLICATIONS

1-S .M. Mostafa, R. Omar, A.abd -eldayem

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February 5,2015*(3085-3097)*

2- Samy M. Mostafa ; A.abd-eldayem

Fuzzy derivations KU-ideals on KU-algebras

eprint arXiv:1506.04475 ,Published in Jun 2015 (1-21)

3- Samy M. Mostafa ; A.abd-eldayem

Q-fuzzy derivations KU-ideals on KU-algebras

Algebra Letters,septemper1,2015(1-18)

Chapter 1

Basic concepts and results

In this chapter we collect all the necessary preliminaries which will be useful in our discussions in the main part of the thesis.

§ (1.1) Introduction to BCK – algebras

In this section, we review some definitions and results that are needed without proofs.

These results in this section are taken from [23,24,25,26,31,32,42,56,58]

Preliminaries

Definition 1.1.1[24] : Let X be a set with a binary operation " * " and a constant 0,then $(X,*,0)$ is called a BCI -algebra ,if it satisfies the following axioms:

(BCI -1) $((x*y)*(x*z))*(z*y)=0$

(BCI -2) $(x*(x*y))*y=0$

(BCI -3) $x*x=0$

(BCI -4) $x*y=0\, and\ y*x=0\ implies\, x=y$

for all $x,y,z\in X$

If a BCI -algebra X satisfies the identity $0*x=0$, for all $x\in X$, then X is called a BCK algebra. It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. For brevity X is called a BCK - algebra .

A binary relation $\leq$ in X is defined by : x ≤ y if and only if $x*y=0$,then by using this relation we can write the definition of BCK algebra as follows :

$(X,*,0)$ is a BCK - algebra if and only if it satisfies that :

(BCI `$_1$) : $((x*y)*(x*z))\leq z*y$

(BCI `$_2$) : $(x*(x*y))\leq y$

(BCI `3) : $x \le x$,

(BCI `4) $x \le y and\ y \le x implies\ x = y$

(BCI `5) : $0 \le x$.

In a BCK - algebra $(X,*,0)$, the following properties are satisfied :

1. $x \le y$ *implies* $z * x \le z * y$
2. $x \le y$ *and* $y \le z$ *imply* $x \le z$
3. $(x * y) * z = (x * z) * y$
4. $(x * y) \le z$ *implies* $x * z \le y$
5. $(x * z) * (y * z) \le x * y$.
6. $x \le y$ *implies* $x * z \le y * z.$
7. $(x * (x * y)) * (y * x) \le x * (x * (y * (y * x))))$
8. $x * y \le x$
9. $x * 0 = x$ for all $x, y, z \in X$.

Example 1.1.2 : Let $X = \{0,1,2,3,4\}$ in which * is defined by the following table :

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	1	1	1	0

Then X is BCK - algebra .

Theorem 1.1.3 [24] : An algebra $(X,*,0)$ of type (2,0) is a BCK - algebra if and only if it statisfies the following conditions :

(BCI$_1$) : $((x * y) * (x * z)) * (z * y) = 0$ and $x * (0 * y) = x$

(BCI4) : $x * y = 0 = y * x$ implies $x = y$, for all $x, y, z \in X$.

For any $x, y \in X$ denote $x \wedge y = y * (y * x)$

Obviously $x \wedge x = x,\ x \wedge 0 = 0 \wedge x = 0$. But in general , $x \wedge y \neq y \wedge x$

Definition 1.1.4 [24] : Let $(X,*,0)$ be a BCK – algebra , and let S be a non – empty subset of X , then S is called a sub - algebra of X , if for all $x, y \in S,\ x * y \in S$, i.e S is closed under the binary operation $*$ of X .

Example 1.1.5 : Let $X = \{0,1,2\}$ in which $*$ is defined by the following table :

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $S = \{0,2\}$ is BCK sub-algebra of X .

Theorem1.1.6 [24] : Suppose that $(X,*,0)$ is a BCK - algebra and let S be a sub - algebra of X : then :

(i) $0 \in S$

(ii) $(S,*,0)$ is also a BCK – algebra

(iii) X is a subalgebra of X

(iv) $\{0\}$ is also a subalgebra of X .

§ *1.2 Bounded BCK-algebras*

The main results in this section are taken from [23]

Definition 1.2.1 : If there is an element 1 of a BCK – algebra X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is called unit of X . A BCK – algebra with unit is said to be bounded .

Example 1.2.2 : Let $X = \{0,1,2,3,4\}$ in which * is defined by the following table :

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X,*,0)$ is a bounded BCK-algebra with unit 4 .

Note : In a bounded BCK – algebra X , we denote $1*x$ *by* N_x .

Definition1.2.3 : For a bounded BCK – algebra X , if an element x satisfies $NN_x = x$, then x is called an involution .

Theorem 1.2.4 In a bounded BCK – algebra X , we have :

(a) $N_1 = 0$, $N_0 = 1$,

(b) $NN_x \le x$

(c) $N_y * N_x \le y * x$

(d) $y \le x$ *implies* $N_x \le N_y$

(e) $N_{x*y} = N_{y*x}$

(f) $NNN_x = N_x$

Theorem 1.2.5 : In a bounded BCK – algebra X , we have $x * N_y = y * N_x$,

for all x *and* $y \in S(X)$ where $S(X)$ is the set of all involutions of a bounded BCK- algebra .

Note that : $NN_0 = N_1 = 0$, and $NN_1 = N_0 = 1$, then the elements 0 and 1 are contained in $S(X)$. Hence $S(X)$ is non - empty .

Theorem 1.2.6 : For any bounded BCK - algebra X , we have $S(X)$ is a bounded sub-algebra of X .

Theorem1.2.7: A BCI –algebra X satisfying $x*(y*z)=(x*y)*z$ is a group in which every element is an involution .

§ *1.3 BCK-ideals*

The main results in this section are taken from [27]

Definition 1.3.1 : A non - empty subset I of a BCK-algebra X is called an BCK ideal of X, if it satisfies the following conditions :

(1) $0 \in I$

(2) $x*y \in I$, $y \in I$ implies $x \in I$, for all $x, y \in X$.

Example1.3.2: Let $X=\{0,1,2\}$ in which $*$ is defined by the following table :

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X,*,0)$ is BCK-algebra , and $\{0\}$, X , $\{0,1\}$, $\{0,2\}$ are all ideals of X .

Example 1.3.3: Let $X=\{0,1,2,3\}$ in which $*$ is defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Then $(X,*,0)$ is BCK-algebra , it has only two ideals $\{0\}$ and X .

The following results gives some properties of ideals In BCK algebra .

Theorem 1.3.4: Suppose I is an ideal of a BCK-algebra X and $x \in I$, if $y \le x$ then $y \in I$.

Definition 1.3.5 Given a BK-algebra $(X,*,0)$ and given elements a,b of X ,we define $A(a,b)=\{x \in X : x*a \le b\}$.

Theorem 1.3.6: Suppose I is a non-empty subset of a BCK-algebra X , then I is an ideal of X if and only if , for any $x,y \in I$, $A(x,y) \subseteq I$.

Corollary 1.3.7: I is an ideal of X if and only if for all $x,y \in I$, $(z*x)*y=0$ implies $z \in I$

§ *1.4.Homomorphisms and isomorphisms on BCK/ BCI -algebras*

The main results in this section are taken from K.Iseki [25] , Y.B.Jun [30] and [41] .

Definition 1.4.1: Let $(X,*,0)$ and $(X',*',0)$ are two BCK-algebras . A mapping $f : X \to X'$ is called a homomorphism from X into X' if , $\forall x,y \in X, f(x*y)=f(x)*'f(y)$. If f is onto i.e $f(X)=X'$, then f is called an epimorphism . If f is both an epimorphism and one to one , then f is called an isomorphism .

Denote the set of all homomorphisms from X *to* X' by Hom (X , X') , this set is always non empty since it contains the zero hom. $0 : X \to X'$, which is defined by $0*x=x'$ $\forall x \in X$. Let $f \in Hom(X,X')$ and $(B \subseteq X')$ The set $\{x \in X : f(x) \in B\}$ is denoted by $f^{-1}(B)$ and is called the inverse image of B under f . In particular , $f^{-1}(\{0'\})=\{x \in X : f(x)=0'\}$ is called the kernel of f . In case $X=X'$, a homomorphism is called an endomorphism and isomorphism is referred as an automorphism . The identity mapping $1 : X \to X'$ is clearly an endomorphism .

Theorem 1.4.2: Suppose $f : X \to X'$ is an homomophism , then :

(a) $f(0)=0'$,

(b) f is an isotone , i.e if $x \le y \Rightarrow f(x) \le f(y)$

Theorem 1.4.3 : Let $(X,*,0)$ and $(X',*,0)$ be two BCK-algebras and let B be an ideal of X' then for any $f \in Hom(X,X')$, $f^{-1}(B)$ is an ideal of X .

Corollary 1.4.4 : $Ker(f)$ is an ideal of X .

Theorem 1.4.5 : Suppose I be a proper ideal of a BCK-algebra X , then for any BCK-algebra X' there exists $f \in Hom(X, X')$ such that $Ker(f) = I$ if and only if I is obstinate .

Theorem 1.4.6 : Suppose $X, Y\ and\ Z$ are BCK-algebras , let $h : X \to Y$ be an epimorphism and $g \in Hom(X, Z)$, if $Ker(h) \subseteq Ker(g)$ then there exists a unique homomorphism $f : X \to Z$ such that $f \circ h = g$

Definition 1.4.7 : Given two BCK-algebras X and X'. If there exists an epimorphism $f : X \to X'$ then we call X to be homomorphic to X' , written $X \sim X'$.If there exists an isomorphism $f : X \to X'$ then we call X to be isomorphic to X' ,written $X \cong X'$.

Theorem 1.4.8: Given a BCK-algebras X . If I is an ideal of X ,then the quotient algebra X / I is a homomorphic image of X .

Theorem 1.4.9: Given two BCK-algebras X and Y. If $f : X \to Y$ is an epimorphism , then $X / Ker(f) \cong Y$.

Definition 1.4.10 A mapping f of a BCI-algebra X into itself is called an endomorphism of X if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(0) = 0$. Especially, f is monic if for any $x, y \in X$, $f(x) = f(y)$ implies that $x = y$.

For a BCI-algebra X, denote by X_+ (resp., $G(X)$) the BCK-part (resp., the BCI-G part) of X, that is, $X_+ = \{x \in X / 0 \le x\}$ (resp., $G(X) = \{x \in X / 0 * x = x\}$. Note that $G(X) \cap X_+ = \{0\}$. If $X_+ = \{0\}$,then X is called a p-semisimple BCI-algebra.

Theorem 1.4.11 In a *p*-semisimple BCI-algebra X , the following hold:

(1) $(x * z) * (y * z) = x * y$

(2) $0 * (0 * x) = x$

(3) $x * (0 * y) = y * (0 * x)$

(4) $x * y = 0$ *implies* $x = y$

(5) $x * a = x * b$ *implies* $a = b$

(6) $a * x = b * x$ *implies* $a = b$