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Summary

The concept of soft sets was first introduced by Molodtsov [69] in 1999 as a general mathematical tool for dealing with uncertain objects. In [69, 70], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability and theory of measurement. After presentation of the operations on soft sets [65], the properties and applications of soft set theory have been studied increasingly [7, 57, 70, 77]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [18].

Recently, in 2011, Shabir et al. [84] initiated the study of soft topological spaces. They defined soft topology τ on the collection of soft sets over X . Consequently, they defined basic notions of soft topological spaces such as open soft sets, closed soft sets, soft subspace, soft closure, soft nbd of a point, soft regular spaces, soft normal spaces and established their several properties. Hussain et al. [35] investigated the properties of open (closed) soft sets, soft nbd and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology.

The notion of fuzzy set was introduced by Zadeh [94] in 1965. Three years later, Chang [19] gave the definition of fuzzy topology. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [5, 6, 11, 16, 63, 64, 65, 66, 70, 71, 96].

Again, in [13], the notion of fuzzy soft set was introduced as a generalization of soft set and some of its basic properties had discussed in detail. Tanay et al. [88] introduced the notion of fuzzy soft topology

over a subset of the initial universe set, while Roy et al. [82] gave the definition of fuzzy soft topology over the initial universe set. In [3], Karal et al. defined the notion of a mapping on classes of fuzzy soft sets and studied its properties.

The main aims of this thesis can be summarized, as follows:

- Introducing some types of open soft sets in soft topological spaces, study their basic properties and the relations between them.
- Introducing the notion of supra soft topological spaces and extending some topological properties to such spaces.
- Introducing the notion of soft ideals in soft set theory to find new soft topologies finer than the original one and generalize some topological properties to such spaces.
- Extending some topological properties, like soft separation axioms, soft compactness and soft connectedness to soft topological spaces via soft ideals.
- Introducing some types of soft continuity in soft topological spaces (via soft ideals) , study their basic properties and the relations between them.
- Extending some topological properties, like soft separation axioms and soft connectedness to fuzzy soft topological spaces.

This thesis contains six chapters, as follows:

Chapter 1: Initiate generalization, providing the reader with results concerning, topological spaces, fuzzy topological spaces, ideals, soft topological spaces and fuzzy soft topological spaces.

Chapter 2: The main purpose of chapter 2, is to introduce the notions of γ -operation, pre open soft sets, α -open sets, semi open soft sets and β -open soft sets in soft topological spaces. We study the relations between these types of open soft sets.

Also, we introduce the notion of supra soft topological spaces. Moreover, we extend the notion of γ -operation, pre open soft sets, α -open soft sets, semi open soft sets, β -open soft sets to such spaces and study their properties and the relations between them. Finally, we introduce the concept of supra generalized closed soft sets in a supra topological space (X, μ, E) and study their properties in detail.

Some results of this chapter are:

- S. A. El-Sheikh and A. M. Abd El-latif, Decompositions of some types of supra soft sets and soft continuity, International Journal of Mathematics Trends and Technology, 9 (1) (2014) 37-56. [24]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ -operation and decompositions of some forms of soft continuity in soft topological spaces, Ann. Fuzzy Math. Inform., 7 (2) (2014) 181-196. [42]
- Kandil 13thA.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Notes on γ -soft operator and some counter examples on (supra) soft continuity, Ann. Fuzzy Math. Inform., 10 (2015). [44]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Supra generalized closed soft sets with respect to soft ideal in supra soft topological spaces, Appl. Math. Inf. Sci., 8 (4) (2014) 1731-1740.[52]

Chapter 3: In this chapter, we introduce the notion of soft ideal in soft set theory. The concept of soft local function is also introduced. These concepts are discussed with a view to find new soft topologies from the original one. The basic structure, especially a basis for such generated soft topologies also studied here. The notion of compatibility of soft ideals with soft topologies is introduced and some equivalent conditions concerning this topic are established here.

Also, the notions of \tilde{I} - (resp. pre- \tilde{I} -, α - \tilde{I} -, semi- \tilde{I} -, β - \tilde{I} -, quasi- \tilde{I} -) open soft sets are introduced. The relations between these types of open soft sets have obtained. Finally, we extend the notions of supra generalized closed soft sets via soft ideals in supra topological space (X, μ, E) and some of their basic properties have studied.

Some results of this chapter are:

- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ -operation and decompositions of some forms of soft continuity of soft topological spaces via soft ideal, Ann. Fuzzy Math. Inform., 9 (3) 385-402. [43]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft ideal theory, soft local function and generated soft topological spaces, Appl. Math. Inf. Sci., 8 (4) (2014) 1595-1603. [46]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Supra generalized closed soft sets with respect to soft ideal in supra soft topological spaces, Appl. Math. Inf. Sci., 8 (4) (2014) 1731-1740. [52]

Chapter 4: In this chapter, we introduce new soft separation axioms based on the semi open soft sets and the soft ideal, which are more general than that's based on the open soft sets. We show that, the properties of soft semi T_i -spaces ($i = 1, 2$) are soft topological properties under the bijection and irresolute open soft mapping. Also, the property of being soft semi regular and soft semi normal are soft topological properties under a bijection, irresolute soft and irresolute open soft functions. Further, we show that the properties of being soft semi T_i -spaces ($i = 1, 2, 3, 4$) are hereditary properties.

Moreover, we introduce new types of soft compactness based on the soft ideal notion in a soft topological space (X, τ, E) namely, soft \tilde{I} -compactness, soft semi- \tilde{I} -compactness, soft countably- \tilde{I} -compactness and soft countably semi- \tilde{I} -compactness and some of their properties have investigated.

Finally, we introduce the notion of connectedness in soft topological spaces based on the soft ideal and study their properties in detail.

Some results of this chapter are:

- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft connectedness via soft ideals, Journal of New Results in Science, 4 (2014) 90-108. [45]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi compactness via soft ideals, Appl. Math. Inf. Sci., 8 (5) (2014) 2297-2306. [47]

- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi (quasi) Hausdorff spaces via soft ideals, South Asian J. Math., 4 (6) (2014) 265-284. [48]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft (semi) regularity (normality) via soft ideals, Appl. Math. Inf. Sci. Lett., 3 (2) (2015) 1-9. [49]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi separation axioms and irresolute soft functions, Ann. Fuzzy Math. Inform., 8 (2) (2014) 305-318. [50]

Chapter 5: In this chapter, we introduce the concepts of pre-continuous soft, α -continuous soft, semi-continuous soft, β -continuous soft functions and discuss their properties in detail. We show that, a mapping between two soft topological spaces is α -continuous soft if and only if it is pre-continuous soft and semi continuous soft.

Also, we introduce the concepts of supra pre- (resp. α -, semi-, β -) continuous soft functions and study some of their properties. We introduce the concepts of \tilde{I} - (resp. pre- \tilde{I} -, α - \tilde{I} -, semi- \tilde{I} -, β - \tilde{I} -, quasi- \tilde{I} -) continuous soft functions and discuss their properties in detail. Finally, some of the basic properties of semi- \tilde{I} -irresolute soft functions and quasi- \tilde{I} -irresolute soft functions, as well as the connection between them, are investigated.

Some results of this chapter are:

- S. A. El-Sheikh and A. M. Abd El-latif, Decompositions of some types of supra soft sets and soft continuity, International Journal of Mathematics Trends and Technology, 9 (1) (2014) 37-56. [24]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ -operation and decompositions of some forms of soft continuity in soft topological spaces, Ann. Fuzzy Math. Inform., 7 (2) (2014) 181-196. [42]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ -operation and decompositions of some forms of soft continuity of soft topological spaces via soft ideal, Ann. Fuzzy Math. Inform., 9 (3) 385-402.[43]

- Kandil 13thA.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Notes on γ -soft operator and some counter examples on (supra) soft continuity, Ann. Fuzzy Math. Inform., 10 (2015). [44]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi (quasi) Hausdorff spaces via soft ideals, South Asian J. Math., 4 (6) (2014) 265-284.[48]

Chapter 6: In this chapter, we introduce the notions of fuzzy semi open (closed) soft sets, fuzzy semi soft interior (closure), fuzzy semi continuous (open) soft functions and fuzzy soft semi separation axioms. We study the relationship between fuzzy semi soft interior and fuzzy semi soft closure. Moreover, we study the properties of fuzzy soft semi regular spaces and fuzzy soft semi normal spaces. Also, we investigate the properties of fuzzy soft semi connected sets, fuzzy soft semi separated sets, fuzzy soft semi s -connected sets and several interesting properties have established.

Some results of this chapter are:

- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Fuzzy soft semi connected properties in fuzzy soft topological spaces, Math. Sci. Lett. 4 (2015). [41]
- A.Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Some fuzzy soft topological properties based on fuzzy semi open soft sets, South Asian J. Math., 4 (4) (2014) 154-169. [51]

0.1 Historical perspective

The origin of soft set theory could be traced to the work of Pawlak [75] in 1993. The notion of soft sets is a unified view of classical, rough and fuzzy sets. This motivated Molodtsovs work [69] in 1999, in which the basic notions of the theory of soft sets and some of its possible applications were presented. For positive motivation, the work discusses some problems of the future with regards to the theory.

In order to solidify the theory of soft set, Maji et al. [62] in 2002, defined some basic terms of the theory such as equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, and absolute soft set with examples. Binary operations like AND, OR, union and intersection were also defined. De Morgans laws and a number of results are verified in soft set theory context. Sezgin et al. [83] proved that certain De Morgans law holds in soft set theory with respect to different operations on soft sets. Irfan et al. [36] in 2009, introduced some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. They improved the notion of complementation in soft set and also proved that certain De Morgans law hold in soft set theory.

In [62], soft set theory was applied to solve a decision making problem using rough set theory and an algorithm to select the optimal choice of an object was provided. This algorithm uses fewer parameters to select the optimal object for a decision problem. However, in decision making problem in [62], there is a straightforward relationship between the decision values of objects and the conditional parameters. That is, the decision values is computed with respect to the conditional parameters. This is quite different in the case of rough sets. In rough set theory the decision attributes are not computed according to the conditional attributes.

Babitha et al. [12] in 2010, introduced the concept of soft set relations as a sub soft set of the cartesian product of the soft sets and many related concepts such as equivalent soft set relation, partition, composition and function are discussed. Also, Kharal et al. [4] defined mapping on soft classes and studied several properties of images and inverse images of soft sets supported by examples and counterexamples. These notions were applied to the problem of medical diagnosis

in medical expert systems.

Cagman et al. [16] in 2010, defined soft matrices and their operations to construct a soft max-min decision making method which can be successfully applied to the problems that contain uncertainties.

Chen et al. [20] in 2005, focus their discussion on the parameterization reduction of soft sets and its applications. First they pointed out that the results of soft set reductions offered in [62] are not correct. They also observe that the algorithms used to first compute the choice value to select the optimal objects for the decision problems are not reasonable and they illustrate this with an example. Finally, they proposed parameterization reduction of soft sets and compared it with the concept of attributes reduction in rough sets theory.

Aktas et al. [6] in 2007, extended the basic properties of soft sets in [62] and compared the soft sets to the related concept of fuzzy sets and rough sets. Then, they gave a definition of soft groups, and derived their basic properties. In 2008, Karadogan [54] work on the application of fuzzy soft set theory in the selection of underground mining method. Ahmad et al. [3] in 2009, further contribute to the properties of fuzzy soft sets and support them with examples. They also defined some operations on fuzzy soft set like union and intersection and proved some basic properties.

Feng et al. [27] in 2008, extended the study of soft set to soft semirings. The notions of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms were introduced, and several related properties were investigated. The theory of BCK/BCIalgebras is now being studied via soft set theory [40].

Herawan et al. [34] in 2009, did a comparative study between rough sets and soft sets and presented a direct proof that Pawlaks and Iwikis rough sets can be considered as soft sets.

Mushrif [73] in 2006, studied the texture classification via soft set theory. In 2010, Wei Xu et al.[91] introduced the notion of vague soft set which is an extension of soft set. The basic properties of vague soft sets are presented and discussed.

Qin et al. [79] dealt with algebraic structure of soft sets. The lattice structures of soft sets were constructed. The concept of soft equality

was introduced and some related properties were derived. It is proved that soft equality is a congruence relation with respect to some operations and the soft quotient algebra was introduced. Jun et al.[40] in 2008, applied the notion of soft sets to commutative ideals of BCK-algebras. Implicative soft ideals and commutative soft ideals were also investigated.

Herawanet al.[34] in 2010, gave an alternative approach for attribute reduction in multi-valued information system under soft set theory. The work emphasized that based on the notion of multi-soft sets and AND operation, attribute reduction can be defined. It is shown that the reducts obtained are equivalent with Pawlaks rough reduction.

Cagman et al. [17] in 2010, extended soft set to fuzzy parameterized fuzzy soft sets and explicate their applications indecision making method. Also, in the same year Gong et al. [30], proposed the concept of bijective soft set and some of its operations were discussed. Finally, an application of bijective soft set in decision-making problems was also discussed.

In 2011, Cagman et al. [17] redefined some of the operations of soft sets to make them more functional to improve several new results. By using these new definitions they also construct a decision making method which selects a set of optimum elements from the alternatives. Also, in the same year, X.Ge et al. [29] further investigate the operational rules given by Maji et al. [62] and Irfan et al. [36]. They obtain some necessary and sufficient conditions that made corresponding operational rules to hold. Jun et al.[39] applied the notion of soft sets to the theory of BCK-algebras.

The notions of positive implicative soft ideals and positive implicative idealistic soft BCK-algebra were introduced and their basic properties were derived. Alkhazaleh et al. [8] extended soft set to soft multi set and present its basic operations such as complement, union and intersection.

Ozturk et al. [74] in 2011, illustrate the interconnections between the various operations in soft set and defined the notion of restricted symmetric difference of soft sets and investigated its properties. Also, Manemaran [67] in 2011, further discussed fuzzy soft sets algebraic structures and defined fuzzy soft group. Operations on fuzzy soft