Oscillation Criteria of Functional Nonlinear Dynamic Equations

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Summary

It is well known that, the differential equations fined a wide range of applications in biology, physics, social, engineering and other areas. The fundamental problem in the theory of differential equations is to deduce the qualitative properties of the solutions of a given equation from the analytic form of the equation because the nonintegrability of the equation makes the problem of obtaining solutions of differential equations in terms of the elementary functions of analysis not solvable for most equations. The oscillation theory of differential equations as a part of qualitative properties are very important for applications.

In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of various equations on time-scales. So, it is the focus of our study.

The thesis consists of four chapters:

In Chapter 1, we state some basic definitions and theorems that will be used throughout the next chapters. Also, we introduce the time scales calculus. Finally, we mention some of the related results with our work.

In Chapter 2, we are concerned with the oscillatory behavior of forced second order functional differential equations with γ -Laplacian, damping and mixed nonlinearities on the form of

$$\left(r(t)\phi_{\gamma}(x'(t))\right)' + p(t)\phi_{\gamma}\left(x'(t)\right) + q_0(t)\phi_{\beta}\left(x(t)\right) + \int_a^b q(t,s)\phi_{\alpha(s)}\left(x\left(g(t,s)\right)\right)d\zeta(s) = e(t)$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\gamma, \beta \in [0,\infty), -\infty < a < b \le \infty$, $\alpha \in C[a,b)$ is strictly increasing such that $0 \le \alpha(a) < \mu < \alpha(b-)$ with $\beta > \gamma > \mu > 0$; $r, p, q_0, e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$ on $[t_0,\infty)$; $q \in C([0,\infty)\times[a,b))$; and $\zeta : [a,b) \to \mathbb{R}$ is nondecreasing. The function $g \in C\big([0,\infty)\times[a,b), [0,\infty)\big)$ is such that $\lim_{t\to\infty} g(t,s) = \infty$, for $s \in [a,b)$. Our interest is to establish oscillation criteria for above equation without assuming that $p(t)$, $q_0(t)$, $q(t,s)$, and $e(t)$ are definition of sign. Also, illustrative example of our results is presented.

Our Results in this Chapter have been published in Serdica Math. J. 40, (2014), 55-76.

In Chapter 3, we are concerned with the oscillatory behavior of the second order nonlinear functional dynamic equation with γ -Laplacian, damping and nonlinearities given by Riemann-Stieltjes integrals

$$\left(a(t)\phi_\gamma(x^\Delta(t))\right)^\Delta + p(t)\phi_\gamma\left(x^{\Delta\sigma}(t)\right) + q_0(t)\phi_\gamma\left(x(g_0(t)\right) + \sum_{i=1}^{2}\int_{a_i}^{b_i} q_i(t,s)\phi_{\alpha_i(s)}\left(x\left(g_i(t,s)\right)\right)\Delta\zeta_i(s) = 0,$$

on a time scale $\mathbb{T}$ which is unbounded above, where $\phi_\gamma(u) := |u|^{\gamma-1}u$, $\gamma > 0$ and for $i = 1,2$, $\alpha_i \in C(\hat{\mathbb{T}},\mathbb{R})$ is strictly increasing with $-\infty < a_i < b_i < \infty$ and $\hat{\mathbb{T}}$ is a time scale; and $\zeta_i \in C(\hat{\mathbb{T}},\mathbb{R})$ is nondecreasing; a and p are positive $rd-$continuous functions on $\mathbb{T}$; q_0 and q_i, $i = 1,2,$ are nonnegative $rd-$continuous functions on $\mathbb{T}$ and $\mathbb{T}\times\hat{\mathbb{T}}$ with $q_0, q_i \not\equiv 0$; the functions $g_0 : \mathbb{T}\to\mathbb{T}$ and $g_i : \mathbb{T}\times\hat{\mathbb{T}}\to\mathbb{T}$ are $rd-$continuous functions such that $\lim_{t\to\infty} g_0(t) = \infty$ and $\lim_{t\to\infty} g_i(t,s) = \infty$ for $s \in \hat{\mathbb{T}}$.

Both of the two cases

$\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(t)\Delta t = \infty$ and $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(t)\Delta t < \infty$, are considered.

The obtained results in this Chapter have been published in Mediterr. J. Math., 13, (2016), 981-1003.

In Chapter 4, we are concerned with the oscillation of solutions of third order nonlinear functional neutral dynamic equation of the form

$$\left\{r_2(t)\phi_{\alpha_2}\left(\left[r_1(t)\phi_{\alpha_1}(z^{\Delta}(t))\right]^{\Delta}\right)\right\}^{\Delta} + q(t)\phi_{\beta}(x(g(t))) = 0,$$

on an above-unbounded time scale $\mathbb{T}$, where $z(t) := x(t) \pm p(t)x(\tau(t))$; $\phi_\alpha(s) := |s|^{\alpha-1}s$, $\alpha_1, \alpha_2, \beta > 0$; p, q, r_i, $i = 1,2$, are positive $rd-$continuous functions on $\mathbb{T}$ with $q(t) \neq 0$ such that $0 \leq p(t) \leq \bar{p} < 1$.

Our results extended and improve some known results for oscillation of third order nonlinear functional neutral dynamic equation. Some our results are illustrated by examples.

Chapter 1

Chapter 1

Preliminary Results for Oscillation of the Dynamic Equations

1.1 Introduction

It is well known that the differential equations fined a wide range of applications in biology, physics, social, engineering and other areas. The fundamental problem in the theory of differential equations is to deduce the qualitative properties of the solutions of a given equation from the analytic form of the equation because the nonintegrability of the equation makes the problem of obtaining solutions of differential equations in terms of the elementary functions of analysis not solvable for most equations. The oscillation theory of differential equations as a part of qualitative properties are very important for applications. For the basic theory of these equations, we refer the reader to the books by K. Gopalsamy [31], I. Győri and G. Ladas [34], Hale and Verduyn Lunel [41] and Kuang [51]. The oscillation theory for delay differential equations has been studied by many authors. An account of these results can be found in [7], [18], [32] and reference there in. The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger (1988) [42], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice—once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the real. The most three popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus. Dynamic equations on a time scale have enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [1] and the references cited therein. A book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [8] summarizes and organizes much of time scale calculus. For advances of dynamic equations on time scales we refer the reader to the book [9]. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of various equations on time-scales; we refer the reader to the papers [2] and [79].

This Chapter is organized to seven sections as follows: after this introduction, in section 2. Time Scales Calculus. In section 3, Exponential functions. In section 4, Interval Criteria for Forced Oscillation of Differentail Equations. In section 5, Oscillation Criteria for Second Order Dynamic Equations. In section 6, Oscillation Criteria for Third Order Dynamic Equations. Finally, in section 7, Oscillation criteria for third order neutral dynamic equations.

1.2 Time Scales Calculus

In this section, we state some basic definitions and theorems that will be used throughout the next chapters.

Definition 1.2.1 *A time scale is an arbitrary nonempty closed subset of the real numbers* $\mathbb{R}$*.*

Thus

$$\mathbb{R},\ \mathbb{Z},\ \mathbb{N},\ \mathbb{N}_0,$$

i.e., the real numbers, the integers, the natural numbers and the nonnegative integers are examples of time scales, as are

$$[0,1]\cup[2,3],\ [0,1]\cup\mathbb{N},\ \text{and the Cantor set.}$$

while

$$\mathbb{Q},\ \mathbb{R}/\mathbb{Q},\ \mathbb{C},\ (0,1),$$

i.e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1 are not time scales. Throughout this thesis, a time scales is denoted by the symbol $\mathbb{T}$ and has the topology that it inherits from the real numbers with the standard topology.

Definition 1.2.2 *For* $t \in \mathbb{T}$*, the forward operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$