بسم الله الرحمن الرحيم

شبكة المعلومات الجامعية

شبكة المعلومـــات الجامعية

التوثيق الالكتروني والميكروفيلم

شبكة المعلومات الجامعية

جامعة عين شمس

التوثيق الالكتروني والميكروفيلم

قسم

نقسم بالله العظيم أن المادة التي تم توثيقها وتسجيلها علي هذه الأفلام قد اعدت دون آية تغيرات

يجب أن

تحفظ هذه الأفلام بعيداً عن الغبار

في درجة حرارة من 15 – 20 مئوية ورطوبة نسبية من 20-40 %

To be kept away from dust in dry cool place of 15 – 25c and relative humidity 20-40 %

شبكة المعلومات الجامعية

بعض الوثـــائق

الأصـلية تالفــة

بالرسالة صفحات

لم ترد بالأصــــل

Topics in Algebraic Logic

A Thesis Presented

By

Tarek Mohamed Abbas Sayed Ahmed

Department of Mathematics,
Faculty of Science, Cairo University

For the Degree of Doctor of Philosophy in Mathematics

TO

Department of Mathematics,
Faculty of Science
Cairo University

Supervisors

Prof. Dr. Ismail Amin	Prof. Dr. Mohamed Amer	Prof. Dr. Istvan Ne'meti
Cairo University	Cairo University	Hungarian Academy of Science

November 2002

Prof. Dr. Ahmed Ghaleb,
The head of Department
of Mathematics,
Faculty of Science, Cairo University

APPROVAL SHEET

Title of the Ph. D Thesis

Topics in Algebraic Logic

Name of the candidate

Tarek Mohamed Abbas Sayed Ahmed

Submitted to:

Faculty of Science- Cairo University

Supervision Committee:

Prof. Dr. Ismail Amin,
Department of Mathematics,
Faculty of Science,
Cairo University.

Prof. Dr. Mohamed Amer,
Department of Mathematics,
Faculty of Science,
Cairo University.

Prof. Dr. Istvan Ne'meti,
Mathematical Institute,
Hungarian Academy of Science,
Hungary.

Prof.Dr. Y.M. Issa
Vice Dean For Post Graduate Studies
Faculty Of Science
Cairo University

Prof. Dr. Ahmed Ghaleb,
The head of Dept. of Mathematics,
Faculty of Science, Cairo University.

Addresses

The supervisors

- Prof. Dr. Ismail Amin
  Department of Mathematics
  Faculty of Science
  Cairo University
  Giza, Egypt

- Prof. Dr. Mohamed Amer
  Department of Mathematics
  Faculty of Science
  Cairo University
  Giza, Egypt
  email:aamer@math-sci.cairo.eun.eg

- Prof Dr. Istvan Nemeti
  Department of Mathematics
  Mathematical Institute
  Hangerian Academy of Science
  Budapest, Hungary
  email:Nemeti@renyi.hu

The author

- Tarek Sayed Ahmed
  Department of Mathematics
  Faculty of Science
  Cairo University
  Giza, Egypt
  email: rutahmed@rusys.eg.net

Contents

Summary

This dissertation revolves around the notion of neat reducts and the related notion of neat embeddings. It has six chapters. Chapter one is a broad introduction to algebraic logic with emphasis on the significance of the notion of neat reducts. Every other chapter is preceded by an abstract and a more detailed technical preface, and is self-contained. In particular, every chapter can be read independently. In chapters two and three we relate results on neat embeddings to results on amalgamation. In chapter four we address amalgamation proper. In chapter five we relate results on neat embeddings to the algebraic notion of complete representations and to the metalogical one of omitting types. We also solve a long-standing open problem of Tarski and his co-athours Andréka, Henkin, Monk, and Németi on neat reducts but only for the finite dimensional case. In chapter six, we extend this result to the infinite dimensional case.

In more detail, in chapter two we show that the class of α-dimensional neat reducts of β-dimensional algebras in several cylindric-like algebras of relations is not closed under forming subalgebras for any pair of ordinals $1 < \alpha < \beta$. As a corollary we settle a conjecture of Tarski for cylindric algebras in the affirmative. The algebras we deal with in chapter two are Tarski's cylindric algebras, Halmos' quasipolyadic algebras with and without equality and Pinter's substitution algebras. In particular, all algebras considered are reducts of Halmos' polyadic algebras. We give a constrasting result for Halmos' polyadic algebras of infinite dimension; for these we show that the class of neat reducts forms a variety. We also present a property of commutativity of two operators on algebras - that of taking neat reducts with that of forming subalgebras - as a definability condition related to the well-known definability and interpolation properties of Beth and Craig in the corresponding logics. We give a

rather general construction showing that several subclasses of the representable algebras fail to have the strong amalgamation property. Finally, we present some positive results on amalgamation answering questions posed by Pigozzi in his landmark paper [115] on amalgamation published in Algebra Universalis in 1971.

In chapter three we show that several (other) reducts of Halmos' polyadic algebras, the so-called G polyadic algebras, have the strong amalgamation property. Here G is a semigroup of transformations on the set ω of non-negative integers, that specifies the signature of the ω-dimensional algebras in question. This will be done by abstracting away properties of G, the resulting (abstract) semigroup we call strongly rich. We present a concrete G that is strongly rich. We infer that the (well-known) definability theorems of Beth, Craig and Robinson hold for certain extensions of first order logic without equality. The question remains as to whether the strongly rich semigroup G can be further chosen to be finitely presented. This result is relevant to the finitization problem in algebraic logic; which (very roughly) asks for a finitely axiomatizable quasivariety that is an adequate algebraization of first order logic.

In chapter four we review, and in the process unify, two techniques (due to Németi and Pigozzi) for proving results concerning strong amalgamation in algebraic logic. Presenting them in a functorial context as adjoint situations, we show that both techniques can be seen as instances of the use of the Shelah-Keisler ultrapower Theorem in proving Robinson's joint consistency theorem.

In chapter five we give a new characterization of the class of the so-called completely representable cylindric and relation algebras via special neat embeddings. We formulate and prove a statement about representing cylindric algebras that turns out to be equivalent (in ZF) (Zermelo-Frankeal set theory without choice) to the famous Martin's axiom; thus is also independent from ZFC(Zermelo Frankeal set theory with choice) + $\neg CH$ ($\neg CH$ abbreviates the negation of the continuum hypothesis.) Then we relate our results omitting types for finite variable fragments of first order logic. Using examples that abound in the literature of algebraic logic but oriented for other purposes, we show that the Henkin-Orey omitting types theorem. fails for the finite variable fragments of first order logic. We discuss what implications our result has concerning omitting types for (usual) first order logic. We use a Fraïssé-type construction to prove that the class of α neat reducts of β dimensional cylindric algebras for $1 < \alpha < \beta$, or $Nr_\alpha CA_\beta$ for short, when α is finite is not closed under *elementary* subalgebras hence is not elementary. This solves (the finite dimensional part) of a long-standing open problem (the only one remaining) posed as problem 4.4 in the monograph on cylindric algebras by Henkin, Monk and Tarski.

In the final chapter, chapter six, we extend our result in chapter five concerning neat reducts to the infinite dimensional case. We prove that for any pair of infinite ordinals $\alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not elementary. Because an analogous Fraïssé theorem is missing; we construct our desired model in a standard step-by-step argument, which can be seen as an infinite analogue of Fraisse's Theorem for certain infinitary logics, the so-called finitary logics of infinitary relations.

Acknowledgments

I am indebted to my supervisor Prof Dr Istvan Németi for pointing out to me the significance of the notion of neat reducts in algebraic logic. I express my gratitude for Prof Dr. Ismail Amin and Prof Dr. Mohamed Amer for their moral and scientific support throughout this work. Special thanks are due to Hajnal Andréka, Gábor Sági, Judit Madarász and Ildikó Sain for extremely fruitful discussions. I am deeply indebted to Professor Hajnal Andréka for giving me so much of her precious time. I also wish to express my deepest gratitude to the London group, particularly Ian Hodkinson and Robin Hirsch for making my visit to Imperial College possible. Thanks are due to Ian Hodkinson for a major contribution to a proof of one of the main results herein - namely that the class of neat reducts is not elementary. Finally, I would like to extend my general thanks to all algebraic logicians, both declared and latent whose results I have used so freely.

Chapter 1

Introduction

When the ideas thrown out by Boole shall have borne their full fruit, algebra though founded on the idea of number in the first instance will appear as a sectional model of the whole form of thought. Its forms considered apart from their matter will be seen to contain all the forms of thought in general. The anti-mathematical logician says that it makes thought a branch of algebra, instead of algebra a branch of thought. It makes nothing. It finds and it finds the laws of thought symbolized in the forms of algebra.

De Morgan in [37].

Pure mathematics was discovered by Boole, in a work which he called the Laws of Thought. His book was in fact concerned with formal logic and this is the same thing as mathematics.

Bertrand Russell as qouted in [25].

Algebraic logic has arisen as a subdiscipline of algebra mirroring constructions and theorems of mathematical logic. It is similar in this respect to such fields as algebraic goemetry and algebraic topology, where the main constructions and theorems are algebraic in nature, but the main intuitions underlying them are respectively goemetric and topological. The main intuitions underlying algebraic logic are, of course, those of formal logic.

Investigations in algebraic logic can proceed in two conceptually different, but often (and unexpectedly) closely related ways.

First, one tries to investigate the algebraic essence of constructions and results in logic, in the hope of gaining more insight thereby adding to his understanding and his knowledge. Second, one can study certain "particular" algebraic structures (or simply algebras) that arise in the course of his first kind of investigations as objects of interest in their own right and go on to discuss questions which naturally arise independently of any connection with logic. But often such purely algebraic results have impact on the logical counterpart.

In this dissertation, we are concerned with both types of investigations and their unexpected and indeed intriguing interplay and interaction. Our results will mostly address algebraizable (in the sense of [28]) modifications of first order logic, be it extensions or reducts. These are the finite-variable fragments of first order logic, and variants of the so-called finitary logics of infinitary relations studied in the monograph [51] sec 4.3. These logics were extensively studied lately, as multi-modal logics with applications in Computer Science. We will investigate definability and interpolation properties like those of Beth, Craig and Robinson as well as Henkin-Orey's omitting types theorem for these logics. Since the dissertation belongs to the field of algebraic logic, our investigations will of course be algebraic. Our theorems will be formulated (and proved) for the algebraic counterpart of such logics, i.e. for certain classes of algebras. The algebras we deal with are the particular boolean algebras with operators designed particularly to algebraize (such different modifications of) first order logic. These are the cylindric algebras of Tarski and various reducts of Halmos' polyadic algebras.

History and background

In the middle of the nineteenth century George Boole initiated the investigation of a class of algebraic structures which were subsequently called Boolean algebras. The theory of these algebras is directly related to the development of the most elementary part of mathematical logic, namely propositional logic. As is well known however, the theory of Boolean algebras can be developed in a purely algebraic fashion, it has at present numerous connections with several branches of mathematics - (independence results in) set theory, topology and analysis - and hence it can be understood and appreciated by mathematicians unfamiliar with the logical problems to which it owes