Some different treatments of graph labeling

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Contents

Summary

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This thesis consists of five chapters.

Chapter one:

In chapter one we introduce some basic definitions and theorems in number theory and graph theory which we need afterwards.

Chapter two:

In chapter two, we study some necessary conditions for a graph to be prime. Also, we study the dependance of these conditions pairwisely and finally we prove that they are altogether not sufficient.

Chapter three:

In chapter three, we first construct new number theoretic tools. Helping us to formulate a necessary and sufficient condition for a graph to be prime. Second, we state and prove a necessary and sufficient conditions for a graph to be prime.

Chapter four:

In chapter four, we give a procedure to determine whether a graph is prime or not using the results obtained in Chapter 2 and Chapter 3. And finally we apply the procedure on two examples: one of them is prime and the other is non-prime.

Chapter five:

In chapter five, we study some families of prime and non-prime graphs. We conclude by a conjecture on the primality a certain family of graphs and give an example .

The results of Chapters 2,3 and 4 will appear in the international journal Ars Combinatoria.

Abstract

Abstract

"Graph labeling at its heart, is a strong communication between number theory and structure of graphs". Graph labelings were first introduced in the late 1960s. Over the past three decades in excess of 800 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are a few general results on graph labelings. Labeled graphs serve as useful models for a broad range of applications.
We discuss here some necessary and sufficient conditions for a graph to be prime. We also give a necessary and sufficient condition for prime graphs. Finally we give a procedure to determine whether or not a graph is prime.We discuss the primality of some corona graphs and some families of graphs.

Keywords

Graph labeling, prime labeling, the maximal independent subsets of vertices of a graph, partitions of an integer, corona graphs and sum graphs.

Introduction

Introduction

Graph labelings, where the vertices and edges are assigned real values or subsets of a set are subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logical-mathematical). Graph labelings were first introduced in mid sixties. An enormous body of literature has grown around the subject, especially in the last thirty years or so and is still getting embellished due to increasing number of application driven concepts (See [14]).

Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. The qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical theory and energy crisis. Quantitative labelings of graphs have led to quite intricate fields of applications such as Coding Theory problems, including the design of good radar location codes, synch-set codes, missile guidance codes and convolution codes with optimal auto-correlation properties. Labeled graphs have also been applied in determining ambiguities in X-Ray Crystallographic analysis, to design communication network addressing systems, to determine optimal circuit layouts and radio-astronomy, etc. (See [3],[4] and [25]). besides its application in mathematics itself.

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla (see [17]). A graph with vertex set V is said to have a prime labeling if its vertices are labeled with distinct integers 1,2,...,$\mid V \mid$ such that for each edge xy the labels assigned to x and y are relatively prime.

In [22] Seoud and Youssef presented some new families of graphs which have prime labelings. They gave an exact formula for the maximum number of edges in a graph of order n having a prime labeling. They conjectured that all unicycle graphs are prime.

In [33] Youssef gave some necessary conditions for a prime graph . He gave also necessary and sufficient conditions for some disconnected graphs to be prime.

A dual of prime labeling has been introduced by Deretsky, Lee, and

Mitchem (see [8]). They say a graph with edge set E has a vertex prime labeling if its edges can be labeled with distinct integers 1,...,$\mid E \mid$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. They showed that certain families of graphs are vertex prime. They also proved that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjectured that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles.

Here, first, we introduce a sufficient condition for a graph to be prime. Second, we show the independence of some necessary conditions for prime and vertex prime graphs in [33]. Third, we give a new necessary condition for prime and vertex prime graphs. Fourth, we give a necessary and sufficient condition for a graph to be prime. Fifth, we give a procedure to determine whether or not a graph is prime. Sixth, we discuss the primality of some corona graphs $G \odot H$. Seventh, we prove that $K_{n,m}$ is prime if and only if $\min\{m,n\} \leq \pi(m+n) - \pi(\frac{m+n}{2}) + 1$, where $\pi(x) :=\mid \{p : p\ prime, 2 \leq p \leq x\} \mid$. Finally, we conjecture that $K_n \odot \overline{K_m}$ is prime if and only if $n \leq \pi(nm+n)+1$. As an application we give the exact values of n for each $m \leq 20$ for which $K_n \odot \overline{K_m}$ is prime (these computations are made by some Computer Algebra System).
Throughout this thesis, we use the standard notations and conventions in graph theory as in [14] and [15], and in number theory as in [1] and [16]. All graphs here are finite and simple. We use $\mid A \mid$ to denote the size of the set A, i.e., its number of elements. In each graph the dark vertices represents a maximal independent set of vertices whose number is denoted by $\beta(G)$.
Some of the results are accepted for publication in the international journal : "Ars Combinatoria," entitled : "On prime graphs"

Chapter 1

Chapter 1

Background

1.1 Preliminaries

Definition 1.1.1. [16] A positive integer $n \neq 1$ is said to be prime if it has no divisors other than 1 and n and composite if it is not prime.
Theorem 1.1.1. [16] (The fundamental theorem of arithmetic).
The standard form of n is unique; apart from rearrangement of factors, n can be expressed as a product of primes in one way only.
Definition 1.1.2. [16] Let a and b be integers , not both equal to zero, the largest positive integer that divides both a and b is called the greatest common devisor of a and b, and denoted by (a, b). If $(a, b) = 1$ we say that a and b are relatively prime.
Definition 1.1.3. [16] Let x be any real number, then $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.
Remark 1.1.1.
$(k, k + 1) = 1$, for all $k \geq 1$.
Proof.
Let $d = (k, k + 1)$, then $d \mid k$ and $d \mid k + 1$ then $d \mid [(k + 1) - k] = 1$, hence $d = 1$.
Remark 1.1.2.
$(2k + 1, 2k + 3) = 1$, for all $k \geq 0$.
Proof.
Let $d = (2k + 1, 2k + 3)$ then $d \mid 2k + 1$ and $d \mid 2k + 3$ then

$d \mid [(2k+3)-(2k+1)] = 2$ then $d = 1$ or 2 but $2k+1, 2k+3$ are odd numbers then $d \neq 2$ then $d = 1$.

Remark 1.1.3.

$(4k+1, 5k+1) = 1$ for all $k \geq 0$.

Proof.

By division algorithm $5k+1 = 1(4k+1)+k$, $4k+1 = 4(k)+1$, $k = k(1)+0$, then $(4k+1, 5k+1) = 1$.

Remark 1.1.4.

$(m,k) = 1 \Leftrightarrow (m^r, k) = 1$.

Proof.

"$\Rightarrow$" Let $(m,k) = 1$ and suppose to the contrary that $(m^r, k) = d > 1$, then $d \mid k$ and $d \mid m^r$, then $d \mid k$ and $d \mid m$ but $(m,k) = 1$,then $d \mid 1$, which gives a contradiction.

"$\Leftarrow$" let $(m^r, k) = 1$ and suppose to the contrary that $(m,k) = d > 1$, then $d \mid k$ and $d \mid m$ then $d \mid k$ and d$\mid m^r$ but $(m^r, k) = 1$, then $d \mid 1$, which gives a contradiction.

Remark 1.1.5.

$(p_1p_2...p_k, r) = 1 \Leftrightarrow$(p$_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}, r) = 1$, where p_i is a prime number for each i.

Proof.

"$\Rightarrow$" Let $(p_1p_2...p_k, r) = 1$ and suppose to the contrary that $(p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}, r) = d \neq 1$, then $d \mid p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ and $d \mid r$, then there exists a prime p_{i_0}, $i_0 \leq k$ such that $p_{i_0} \mid d$, then $p_{i_0} \mid r$ and $p_{i_0} \mid p_1p_2...p_k$ but $(p_1p_2...p_k, r) = 1$, then $p_{i_0} \mid 1$, which gives a contradiction.

"$\Leftarrow$" Let $(p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}, r) = 1$. Suppose to the contrary that $(p_1p_2...p_k, r) = d > 1$, then $d \mid p_1p_2...p_k$ and $d \mid r$. Hence there exists a prime p_{i_0}, $i_0 \leq k$ such that $p_{i_0} \mid d$, then $p_{i_0} \mid r$. Since $p_{i_0} \mid p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ but $(p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}, r) = 1$, we get $p_{i_0} \mid 1$, which gives a contradiction.

Remark 1.1.6.

If n is a positive integer, then $\sum_{i=0}^{n}(-1)^i C_i^n = 0$.

Proof.

from the binomial theorem $(a+b)^n = \sum_{i=0}^{n} C_i^n a^i b^{n-i}$, then we have $0 = ((-1)+1)^n = \sum_{i=0}^{n} C_i^n (-1)^i 1^{n-i} = \sum_{i=0}^{n}(-1)^i C_i^n$.

1.2 How big is a union of finitely many finite sets ?

Suppose that you have two finite sets A and B. You can find the size of their union using $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$ because when you work out $\mid A \mid + \mid B \mid$ the elements of $A \cap B$ are being 'counted twice'. You compensate for this by subtracting $\mid A \cap B \mid$.
Now suppose you have three finite sets. A very careful analysis of counting will show you that $\mid A \cup B \cup C \mid = \mid A \mid + \mid B \mid + \mid C \mid -$
$\mid A \cap B \mid - \mid A \cap C \mid - \mid B \cap C \mid + \mid A \cap B \cap C \mid$.
What is going on here is that you first try $\mid A \mid + \mid B \mid + \mid C \mid$, but this is wrong because elements in $A \cap B, A \cap C, B \cap C$, have been counted too much.You therefore try to eliminate this over-counting by subtracting $\mid A \cap B \mid + \mid A \cap C \mid + \mid B \cap C \mid$, but then notice that elements of $A \cap B \cap C$, have been 'over removed'.You compensate for this by adding $\mid A \cap B \cap C \mid$ and all is well. We are edging toward the inclusion-exclusion enumeration principle.
Let us look at a concrete example $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{2, 3, 4, 5, 6, 7\}$. Now the previous rule says 7 = 4 + 4 + 6 - 2 - 3 - 4 + 2, which happily is true. We prove the validity of the Inclusion - Exclusion counting principle.

Theorem 1.2.1. (The inclusion - exclusion principle) [16]
Suppose $n \in \mathbb{N}$, and A_i is a finite set for $1 \leq i \leq n$, it follows that $\mid \cup_{1 \leq i \leq n} A_i \mid = \sum_{1 \leq i_1 \leq n} \mid A_{i_1} \mid - \sum_{1 \leq i_1 \leq i_2 \leq n} \mid A_{i_1} \cap A_{i_2} \mid +$
$\sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} \mid A_{i_1} \cap A_{i_2} \cap A_{i_3} \mid -... + (-1)^n \mid \cap_{i=1}^{n} A_i \mid$.
Proof.
The proof will be carried out by induction on $\mid \cup_{1 \leq i \leq n} A_i \mid$, first if all A_i are empty then the theorem holds, and the induction starts without difficulty. Now focus on the case where $\mid \cup_{1 \leq i \leq n} A_i \mid > 0$, pick any $x \in \cup_{1 \leq i \leq n} A_i$, and form new sets by $B_i = A_i \setminus \{x\}$,(i.e., remove x from all the sets A_i which contain it).
Now $\mid \cup_{1 \leq i \leq n} B_i \mid = \mid \cup_{1 \leq i \leq n} A_i \mid -1$, so the theorem holds for the sets B_i by the induction assumption
$\mid \cup_{1 \leq i \leq n} B_i \mid = \sum_{1 \leq i_1 \leq n} \mid B_{i_1} \mid - \sum_{1 \leq i_1 \leq i_2 \leq n} \mid B_{i_1} \cap B_{i_2} \mid$