



Faculty of Education  
Mathematics Department

# **STATISTICAL ANALYSIS FOR ACCELERATED LIFE TESTS BASED ON DIFFERENT CENSORING SCHEMES**

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**(Statistics)**

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# List of Abbreviations and Symbols

- ALT: Accelerated life test.
- PALT: Partially accelerated life test.
- MLEs: Maximum likelihood estimates.
- ML: Maximum likelihood.
- BEs: Bayes estimates.
- PDF: Probability density function.
- CDF: Cumulative distribution function.
- HF: Hazard function.
- HRF: Hazard rate function.
- GP: Geometric process.
- CI: Confidence interval.
- IP: Informative prior.
- NIP: Non-informative prior.
- SE: Square error.
- SEL: SE loss.
- LINEX: Linear exponential.

- LINEXL: LINEX loss.
- MCMC: Markov chain Monte Carlo.
- MTTF: Mean time to failure.
- $Var$ : Variance.
- $AV$ : Asymptotic variance.
- K-S: Kolmogorov-Smirnov.
- AIC: Akaike information criterion.
- TRV: Tampered random variable.
- TFR: Tampered failure rate.
- $EE$ : Extension of the exponential.
- $WE$ : Weibull.
- $GE$ : Generalized exponential.
- $PGW$ : Power generalized Weibull.
- MSEs: Mean square errors.
- RABs: Relative absolute biases.
- CS: Censoring scheme.
- $\mathbb{R}$ : Real numbers.
- $\Phi(\cdot)$ : CDF of the standard normal distribution.
- $E(\cdot)$ : Expected value.
- $U(0, 1)$ : Standard uniform distribution.
- $N(\cdot, \cdot)$ : Normal distribution.
- $<_{st}$ : Less stochastically.

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