بِسْمِ اللهِ الرَّحْمنِ الرَّحِيمِ

(قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ)

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(Pure Mathematics)

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Contents

Summary

The fundamental concept of Efremovič proximity space has been introduced by Efremovič [10]. In addition, Leader [27, 28] and Lodato [29, 30] have worked with weaker axioms than those of Efremovič proximity space enabling them to introduce an arbitrary topology on the underlying set. Furthermore, proximity relations are useful in solving problems based on human perception [40] that arise in areas such as image analysis [20]. The notion of ideal topological spaces was first studied by Kuratowski [26] and Vaidyanathaswamy [63, 64]. Compatibility of the topology with an ideal $\mathcal{I}$ was first defined by Njastad [36]. In 1990, Jankovic and Hamlett [24] investigated further properties of ideal topological spaces.

In 1999, Molodtsov [33] proposed the novel concept of soft set theory, which provides a completely new approach for modeling vagueness and uncertainty. Soft set theory has a rich potential for applications in several directions, few of which were shown by Molodtsov in [33]. After Molodtsov's work, some different applications of soft sets were studied in Chen et al. [8]. Further theoretical aspects of soft sets were explored by Maji et al. [32]. Also the same authors [31] presented the definition of a fuzzy soft set. The algebraic nature of the soft set has been studied by several researchers. Aktas and Cagman [2] initiated soft groups, and Feng [12] defined soft semirings. Babitha and John [3] defined soft set relations and functions. Yang and Guo [67] introduced kernels and closures of soft set relations. Hazra et al. [23] introduced the notion of basic proximity of soft sets. Also, the same authors [22] proposed the notion of soft proximity.

In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset [4] or bag

[66], for short). Thus, a mset differs from a set in the sense that each element has a multiplicity a natural number not necessarily one that indicates how many times it is a member of the mset. One of the most natural and simplest examples is the mset of prime factors of a positive integer n. The number 400 has the factorization $400 = 2^4 5^2$ which gives the mset $\{2, 2, 2, 2, 5, 5\}$. Also, the cubic equation $x^3 - 5x^2 + 3x + 9 = 0$ has roots $3, 3$ and -1 which give the mset $\{3, 3, -1\}$.

Classical set theory is a basic concept to represent various situations in mathematical notation where repeated occurrences of elements are not allowed. But in various circumstances repetition of elements become mandatory to the system. For example, a graph with loops, there are many hydrogen atoms, many water molecules, many strands of identical DNA etc. This leads to effectively three possible relations between any two physical objects; they are different, they are the same but separate, or they are coinciding and identical. For example, ammonia NH_3, with three hydrogen atoms, say H, H and H, and one nitrogen atom, say N. Clearly H and N are different. However H, H and H are the same but separate, while H and H are coinciding and identical. There are many other examples, for instance, carbon dioxide CO_2, sulfuric acid H_2SO_4, and water H_2O etc.

The overall aim of this thesis is to increase the stock of knowledge about proximity relations and some related structures. Specifically we aim to:

- Introduce new approaches of proximity relations based on ideal notion.

- Investigate new approaches of proximity relations based on soft set notion.

- Introduce examples of multiset topologies are not tackled before.

- Extending the notions of compact, proximity relations, proximal neighborhood and proximity mappings to the multiset context.

- Find way to reduce the boundary region of rough sets in the multiset context.

- Study the notion of mild continuity in relator spaces and its properties.

This Thesis includes six chapters as follow:

Chapter 1 has a collection of all basic definitions and notions for further study.

In **Chapter** 2, a new approach of proximity structure based on the ideal notion has been introduced. For $\mathcal{I} = \{\phi\}$, we have the Efremovič proximity structure [10] and for the other types of $\mathcal{I}$, we have many types of proximity structures. Some results on this new approach have obtained and one of the important results: every $*-$normal T_1 space is $\mathcal{I}$-proximizable space (Theorem 2.2.5). Moreover, $\delta_{\mathcal{I}}-$neighborhood in an $\mathcal{I}-$proximity space has been introduced. This provides an alternative description to the study of $\mathcal{I}-$proximity spaces. Furthermore, the operator $*$ on $P(X)$ with respect to an ideal and uniformity $\mathfrak{U}$ on X has been introduced and various properties of it are investigated. The new generated uniformity via ideal is presented which generated a topology $\tau^*(\mathfrak{U})$ finer than the old one. In addition, $\tau^*(\mathfrak{U}) = \tau_{\delta_{\mathcal{I}}}$ is proved. The notion of generalized proximity has been introduced via the concept of ideal in the ordinary topology. In addition to that, the notions of $\mathcal{I}$-Leader, $\mathcal{I}$-Pervin, and $\mathcal{I}$-Lodato proximities have been introduced. It is shown that the relation between the topology generated via these proximities and the topology τ^* which generated via ideal.

It should be noted that some results of this chapter are published as follow:

- A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. Zakaria, *$\mathcal{I}-$proximity spaces*, Jökull Journal **63** (2013), 237-245.
- A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. Zakaria, *Generalized $\mathcal{I}-$proximity spaces*, Mathematical Sciences Letters **3** (2014), 173-178.
- A. Kandil, O. A. Tantawy, S. A. El-Sheikh, A. Zakaria, *On $\mathcal{I}-$proximity spaces*, Journal of Applied Mathematics and Information Science Letters. Accepted.

In **Chapter** 3, a new approaches of proximity and generalized proximity based on the soft set have been introduced. It is shown that every soft T_4- space is compatible with a proximity relation on $P(X)^E$. In addition, every soft space is compatible with a Pervin proximity relation on $P(X)^E$. It is also shown that every soft R_o-space is compatible with a Lodato proximity relation on $P(X)^E$. Furthermore, we introduced a new approach of soft proximity structure based on the ideal notion. For $\mathcal{I} = \{\phi\}$, we have the soft proximity structure [22] and for the other types of $\mathcal{I}$, we have many types of soft proximity structures.

It should be noted that some results of this chapter are published as follow:

- A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. Zakaria, *New structures of proximity spaces*, Information Sciences Letters **3** (2014), 85-89.
- A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. Zakaria, *Soft $\mathcal{I}-$proximity spaces*, Ann. Fuzzy Math. Inform. **9** (2015), 675-682.

Chapter 4, is an attempt to explore the theoretical aspects of msets by extending the notions of compact, proximity relation, proximal neighborhood and proximity mappings to the mset context. Examples of new mset topologies, open msets cover, compact mset and many identities involving the concept mset are introduced. A mset proximity relations and an integral examples of mset proximity are obtained. In addition, a mset topology induced by a mset proximity relation on a mset M and study its principal properties. The concept of mset δ-neighborhood in the mset proximity space which furnishes an alternative approach to the study of mset proximity spaces has been introduced. Furthermore, the mset proximity mappings and its properties are introduced.

It should be noted that some results of this chapter are submitted as follow:

- A. Zakaria, *Note on "on multiset topologies"*. Ann. Fuzzy Math. Inform. **10** (5) (2015), 13-14.
- A. Kandil, O. A. Tantawy, S. A. El-Sheikh and A. Zakaria, *Some structures in multiset context.* Submitted.

Chapter 5 is an attempt to explore a new approach of rough mset to decrease the boundary region and increase the accuracy measure. We show that an alleged properties stated in [16] are invalid in general, by giving a counter-examples. A new approach of ideals in the context of msets on the lattice of all submsets with the order relation as the mset inclusion has been introduced. Properties of mset ideals are studied. The R^*- upper and R_*- lower mset approximations via mset ideals have been mentioned. These definitions are different from Girish et al.'s definitions [16, 18] and more general. If $\mathcal{I} = \{\phi\}$, then our mset approximations coincide with Girish et al.'s mset approximations. So, Girish et al.'s mset approximations are special case of our mset approximations. Moreover, Properties of these mset approximations are studied. Furthermore, an mset topology via this new approach has been introduced. This mset topology is finer than Girish et al.'s one. Moreover, this new approach leads to decrease the boundary region and increase the accuracy measure. In addition, the boundary of a submset decreases as the mset ideal on a nonempty mset M increases. Finally, varied examples are introduced to show the significance of this new approach.

It should be noted that some results of this chapter are published as follow:

- S. J. John, S. A. El-Sheikh and A. Zakaria, *Generalized rough multiset via multiset ideals*, Journal of Intelligent and Fuzzy Systems. Accepted.

- S. A. El-Sheikh and A. Zakaria, *Note on "rough multiset and its multiset topology"*, Ann. Fuzzy Math. Inform. **10** (2015), 235-238.

In **Chapter** 6, Á. Száz and A. Zakaria continue the investigations initiated by Á. Száz [45, 50, 51, 53, 54, 58, 59] on the basic continuity properties of a single relation, and also of a pair of relations, on one relator (generalized uniform) space to another. Furthermore, the notion of mild continuity in relator spaces has been introduced. Moreover, several useful consequences of mild continuity have been mentioned. Finally, the properties of proper, uniform, proximal, topological and paratopological mild continuity have been presented.

It should be noted that some results of this chapter are published as follow:

- Á. Száz and A. Zakaria, *Mild continuity properties of relations*

and relators in relator spaces, Tech. Rep., Inst. Math., Univ. Debrecen 2015/1, 65 pp.

- Á. Száz and A. Zakaria, *Mild continuity properties of relations and relators in relator spaces*, Essays in Mathematics and its Applications: In Honor of Vladimir Arnold. To appear 2016.

Chapter 1

Preliminaries

The purpose of this chapter is to present a short survey of some needed definitions and theories of the material used in this thesis.

1.1 Some basic concepts of topological structures

Definition 1.1.1 *[11] Let X be a nonempty set. A class τ of subsets of X is called a topology on X if τ satisfies the following axioms:*

1. *$X, \emptyset \in \tau$,*
2. *Arbitrary union of members of τ belongs to τ,*
3. *The intersection of any two sets in τ belongs to τ.*

The member of τ is called τ-open set, or simply open set, and the pair (X, τ) is called a topological space . A subset A of a topological space (X, τ) is called a closed set if its complement A^c is an open set.

Definition 1.1.2 *[11] Let τ_1 and τ_2 be two topologies on a set X. τ_1 is said to be finer than τ_2 or τ_2 is said to be coarser than τ_1 if every τ_2-open set is τ_1-open.*

Definition 1.1.3 *[26] Let $P(X)$ be the class of all subsets of X. If $h : P(X) \to P(X)$ is a function satisfying*

1. $h(\emptyset) = \emptyset$,
2. $A \in P(X) \Rightarrow A \subseteq h(A)$,
3. $A, B \in P(X) \Rightarrow h(A \cup B) = h(A) \cup h(B)$, *and*
4. $A \in P(X) \Rightarrow h(h(A)) = h(A)$,

then h is called a Kuratowski's closure operator and the family

$$\tau_h = \{A \in P(X) : h(A^c) = A^c\}$$

is a topology on X.

Theorem 1.1.1 *[26] Let $P(X)$ be the class of all subsets of X. If $K : P(X) \to P(X)$ is a function satisfying*

1. $K(\emptyset) = \emptyset$,
2. $K(A \cup B) = K(A) \cup K(B)$,
3. $K(K(A)) \subseteq K(A)$.

Then $cl : P(X) \to P(X)$ defined by $cl(A) = A \cup K(A)$ is a Kuratowski's closure operator on $P(X)$.

Definition 1.1.4 *[11] Let (X, τ) be a topological space and $\beta \subseteq \tau$. Then β is called a base for the topology τ if it satisfies one of the following conditions:*

1. *Every open set $G \in \tau$ is a union of members of β.*
2. *For any point p belonging to an open set G, there exists $B \in \beta$ with $p \in B \subseteq G$.*

Theorem 1.1.2 *[34] Let $\beta \subseteq P(X)$. β is a topological basis iff the following assertions are satisfied:*

1. $\cup\{B : B \in \beta\} = X$
2. *Given $B_1,\ B_2 \in \beta$ and $x \in B_1 \cap B_2$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.*