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SUMMARY

We are not able to use classical methods to solve some kinds of problems given in sociology, economics, environment, engineering etc., since, these kinds of problems have their own uncertainties. Fuzzy set theory, which was .rstly proposed by Zadeh [121] in 1965, has become a very important tool to solve these kinds of problems and provides an appropriate framework for representing vague concepts by allowing partial membership. Fuzzy set theory has been studied by both mathematicians and computer scientists and many applications of fuzzy set theory have arisen over the years, such as fuzzy control systems, fuzzy automata, fuzzy logic, fuzzy topology etc. Beside this theory, there are also theory of probability, rough set theory which deal with to solve these problems. The notion of fuzzy topology was introduced by Chang [28] . Lowen [81] introduced an other de.nition of fuzzy topology.

The concept of soft sets was .rstly introduced by Molodtsov [91] in 1999 as a general mathematical tool for dealing with uncertain objects. Molodtsov [91], successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability and theory of measurement. After presentation of the operations on soft sets [91], the properties and applications of soft set theory have been studied increasingly [13, 101].

Recently, in 2011, Shabir et al. [108] initiated the study of soft topological spaces. They de.ned soft topology _ on the collection of soft sets over Ξ. Consequently, they de.ned basic notions of soft topological spaces such as open soft sets, closed soft sets, soft subspace, soft closure, soft nbd of a point, soft regular spaces, soft normal spaces and established their several properties. Hussain et al. [55] investigated the

properties of open (closed) soft sets, soft nbd and soft closure. They also de.ned and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology.

Maji et al. [85] initiated the study involving both fuzzy sets and soft sets. In this paper, the notion of fuzzy soft sets was introduced as a fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. Tanay et al. [110] and Simsekler [109] gave the topological structure of fuzzy soft sets and generalized by Chakraborty et al. [27] and Goswami et al. [46]. The fuzzy soft sets have many application such as: making dicision [19, 24, 30, 32, 45, 61, 79, 111] and mobile network [115].

The local properties of a space which may also be in certain cases the properties of the whole space, are important .eld for study in general topology, fuzzy topology, and soft topology. The notion of ideal in general topology was introduced by Kuratowski [81], Vaidyanathaswamy [116, 117] and several other authors carried out such analyses. Recently, there has been an extensive study on the importance of ideal in general topology in the paper of Jankovi´c and Hamlett [58], in fuzzy topology: by Nasef et al. [95], Mahmoud [86] and D. Sarker [107], in soft set theory: by Kandil et. al. [69] in 2014.

The main aims of this thesis can be summarized, as follows:

1- Introducing fuzzy soft ideal theory, fuzzy soft local function and generating a new fuzzy soft topological space by two di¤erent methods.

2- Genaralized fuzzy soft sets and decompositions of some forms of fuzzy soft contiuities via fuzzy soft ideals.

3- Introducing some fuzzy soft topological properties such as: fuzzy soft separation

and regularity axioms, fuzzy soft some classes of compactness in fuzzy soft topological spaces.
iv
4- Introducing some types of fuzzy soft separated sets, some types fuzzy soft connected sets and study the relation between them.
5- Introducing an adiitional types of connectedness in fuzzy soft topological spaces such as: fuzzy soft extremally disconnected spaces, fuzzy soft hyperconnected, fuzzy soft connectedness based on fuzzy soft _-open sets and ?-connectedness in fuzzy soft ideal topological sapces.
This thesis contains six chapters, as follows:
Chapter (I): Initiate generalization, providing the reader with results concerning, fuzzy topological spaces, soft topological spaces, fuzzy soft topological spaces, fuzzy soft point and its neigbourhood structure, fuzzy soft closure, fuzzy soft interior, fuzzy soft accumulation point, fuzzy soft boundary. Also, the represent the notions of fuzzy soft continuity, fuzzy soft separation axioms and fuzzy soft comaptness.
Chapter (II): Our aimof this chapter is to extend those ideal of general topology, fuzzy topology, and soft topology to fuzzy soft setting. In Section 4.1, we de.ne fuzzy soft ideal and introduce the notion of fuzzy soft local function corresponding to a fuzzy soft topological space. We have deduce some characterization theorems for such concepts exactly analogous to general topology, fuzzy topology, and soft topology and succeeded in .nding out the generated new fuzzy soft topologies for any fuzzy topological space. In Section 4.2, we discuss the basic structure of new fuzzy soft topology and it is established that the new fuzzy soft topology cannot be further generated with the same fuzzy soft ideal. Finally, in Section 4.3, we de.ne the local function by using the quasi-coincident relation and study its properties. Also, we introduce the concept of quasi-cover of a fuzzy soft set and introduce the notion of compatibility of fuzzy soft ideal with a fuzzy soft topological space and obtain some results concerning this concept.
Some results of this chapter are:
1- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed,
v
Fuzzy soft ideal topological spaces, South Asian Journal of Mathematics, 6 (4) (2016), 186-198. [65]
2- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Ideal Theory: Fuzzy Soft Local Function and Generated Fuzzy Soft Topological Spaces, The Journal of Fuzzy Mathematics, 25 (2), 2017. [66]
Chapter (III): The purpose of this chapter is to introduce the notions of fuzzy soft semi-eI-open sets (respectively, fuzzy soft eI-open sets, fuzzy soft pre-eI-open sets, fuzzy soft _-eI-open sets, fuzzy soft _-eI-open sets, fuzzy soft almost eI-open, fuzzy soft _-dense-in-itself). Furthermore, we present the notions of fuzzy soft semi-eI-continuous functions (respectively, fuzzy soft eI-continuous functions, fuzzy soft pre-eI-continuous functions, fuzzy soft _-eI-continuous functions, fuzzy soft _-eI-continuous functions, fuzzy soft almost eI-continuous functions, fuzzy soft _-eI-continuous functions) Moreover, the decomposition of such forms of fuzzy soft continuity is studied.
Chapter (IV):The object of this chapter is to inroduce a set of new regularity and separation axioms which are called $(\Phi\Sigma P_{\iota}; \iota = 0; 1; 2; 3)$ and $(\Phi\Sigma T_{\iota}; \iota = 0; 1; 2; 3; 4)$ by using fuzzy soft quasi-coincident and neighborhood system. The notion of fuzzy soft hereditary property is examined. Furthermore, we introduced the ideas fuzzy soft

semi-compactness, fuzzy soft _-compactness, fuzzy soft _-compactness, and fuzzy soft strongly compactness. Also, the notions of fuzzy soft Σ-closed, fuzzy soft σ-closed, fuzzy soft Π-closed, fuzzy soft _-closed and fuzzy soft _-closed are studied. Finite intersection property is used to characterize these concepts. A comparison between these types of compactness in fuzzy soft topological spaces is established.

In Chapter (V): The notions of fuzzy soft connected sets and fuzzy soft connected components are very important in fuzzy soft topological spaces which in turn re.ect the intrinsic nature of it that is in fact its peculiarity. In this Chapter, we introduce some types of fuzzy soft separated sets, some types of connectedness in

fuzzy soft topological spaces and study the relationship between them. Also, we introduce an equivalence relation on fuzzy soft points and de.ne a fuzzy soft connected components as an equivalence class induced by this equivalence relation.

Some results of this chapter are:

1- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces I, Journal of Advances in Mathematics, 12 (8) (2016), 6473-6488. [64]

2- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces II, Journal of Egyptian Mathematical Society, (accepted).

Chapter (VI): In this chapter, we introduce the concept of fuzzy soft extremally disconnected spaces, fuzzy soft Δ-space and fuzzy soft hyperconnected space. The relation between these concepts is investigated. Furthermore, we introduce the notions of fuzzy soft _-separated sets and use it to introduce the notions of fuzzy _-σ-connectness in fuzzy soft topological spaces and study its basic properties. Moreover, we extend the notion of fuzzy soft connectedness via fuzzy soft ideal.

Some results of this chapter are:

1- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Sawsan S. S. El-Sayed, Fuzzy Soft Hyperconnected spaces, Annals of fuzzy mathematics and informatics, (accepted).

2- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, A. M. Abd El-Latif, S. El-Sayed, Fuzzy soft connectedness based on fuzzy _-open soft sets, Journal of Mathematics and Computer Applications Research (JMCAR), 5 (2) (2015), 37-48. [63]

CHAPTER I
PRELIMINARIES

The introductory chapter is considered as a background for the material included in the thesis. The purpose of this chapter is to present a short survey of some needed de.nitions and theories of the material used in this thesis. Also, some concepts related fuzzy soft topological spaces have investigated.

1.1 Fuzzy soft sets and fuzzy soft topological spaces

De.nition 1.1.1. [121] A fuzzy set A of a non-empty set Ξ is characterized by a membership function $\_A$ which associates each point of Ξ to a real number in the interval [0; 1]. With the value of $\_A(\xi)$ at ξ representing the .grade of membership. of ξ in A. Let I^Ξ denotes the family of all fuzzy sets on Ξ . If A; B 2 I^Ξ, then some basic set operations for fuzzy sets are given by Zadeh [121], as follows:

1- A _ B ( ) $\_A(\xi)$ _ $\_B(\xi)$ 8ξ 2 Ξ.

2- A = B ( ) _A(ξ) = _B(ξ) 8ξ 2 Ξ.
3-X = A _ B ( ) _X(ξ) = _A(ξ) _ _B(ξ) 8ξ 2 Ξ.
4-Δ = A ^ B ( ) _X(ξ) = _A(ξ) ^ _B(ξ) 8ξ 2 Ξ.
5-M = A_( ) _M = 1 □ _A(ξ) 8ξ 2 Ξ; A_ is the complement of the fuzzy set A:

De.nition 1.1.2. [28]. Let _ be a collection of fuzzy sets over a universe Ξ, then _
_ IΞ is called a fuzzy topology on Ξ if:

(1) 0, 1 2 _ ,
(2) the fuzzy union of any members of _ belongs to _ ,
(3) the fuzzy intersection of any two members of _ belongs to _ .

The pair (Ξ; _) is called a fuzzy topological space over Ξ. Also, each member of _ is called a fuzzy open in (Ξ; _).

De.nition 1.1.3. [91] Let Ξ be the initial universe set and E be the set of parameters. A pair (Φ;A) ;denoted by ΦA, is called a soft set over Y where Φ is a mapping given by Φ : A ! Π(Ξ) and A _ E.

In other words, the soft set is a parameterized family of subsets of the set Ξ. For ε 2 E , Φ(ε) may be considered as the set of ε-elements of the soft set (Φ;A), or as the set of ε-approximate elements of the soft set. If ε =2 A, then Φ(ε) = _ i.e ΦA = fΦ(ε) : ε 2 A _ E; Φ : A ! Π(Ξ)g. The family of all these soft sets over Ξ is denoted by ΣΣ(Ξ)E.

De.nition 1.1.4. [84] The union of two soft sets (Φ; A) and (Γ; B) over the common universe Ξ is the soft set (H;X) , where X = A [B and for all ε 2 X,

H(ε) =
8>>>><
>>>>:
Φ(ε) ε 2 A □ B
Γ(ε) ε 2 B □ A
Φ(ε) [Γ(ε) ε 2 A \ B

De.nition 1.1.5. [84] The intersection of two soft sets (Φ;A) and (Γ;B) over the common universe Ξ is the soft set (H;X), where X = A \ B and for all ε 2 X, H(ε) = Φ(ε) \ Γ(ε). Note that, in order to e¢ ciently discuss, we consider only soft sets (Φ;E) over a universe Ξ with the same set of parameter E. We denote the family of these soft sets by ΣΣ(Ξ)E.

De.nition 1.1.6. [84] (Φ;E) over a universe Ξ with the set of parameter E is said to be null soft set denoted by e_
if 8ε 2 E, Φ(ε) = _.

De.nition 1.1.7. [84] A soft set (Φ;E) over Ξ with the set of parameter E is said to be absolute soft set denoted by eΞ
if 8ε 2 E, Φ(ε) = Ξ.

De.nition 1.1.8. [84, 101] For two soft sets (Φ;A) and (Γ;B) over the universe Ξ, we say that (Φ;A) is a soft subset of (Γ;B), if

(i) A _ B,
(ii) 8ε 2 A, Φ(ε) _ Γ(ε) and is written as (Φ;A) e_
(Γ;B).
(Φ;A) is said to be soft superset of (Γ;B) if (Γ;B) is a soft subset of (Φ;A) and we write (Φ;A) e_
(Γ;B).
De.nition 1.1.9. [84] The complement of a soft set (Φ;A) is denoted by $(Φ;A)_χ$ and is de.ned by $(Φ;A)_χ$ = $(Φ_χ;A)$, where $Φ_χ$: A ! Π(Y) is a mapping given by $Φ_χ(ε)$ = $[Φ(ε)]_χ$ for all ε 2 A.
De.nition 1.1.10. [108] Let _ be a collection of soft sets over a universe Ξ with a .xed set of parameters E, then _ _ $ΣΣ(Ξ)_E$ is called a soft topology on Ξ if;
1- eΞ
; e_
2 _ , where e_(ε) = _ and eΞ
(ε) = Ξ 8ε 2 E;
2-the union of any number of soft sets in _ belongs to _ ,
3-the intersection of any two soft sets in _ belongs to _ .
The triplet (Ξ; _ ;E) is called a soft topological space over Ξ.
De.nition 1.1.11. [85] Let Ξ be an initial universe set and let E be a set of parameters. Let $I_Ξ$ denotes the collection of all fuzzy subsets of Ξ and A _ E. A pair $(ϕ_A;E)$, denoted by $ϕ_A$, is called a fuzzy soft set over Ξ, if $ϕ_A$ is a mapping given by $ϕ_A$: E ! $I_Ξ$ de.ned by $ϕ_A(ε)$ = $_^ε$
$_{ϕ_A}$, where $_^ε$
$_{ϕ_A}$ = 0 if ε =2 A and $_^ε$
$_{ϕ_A}$ 6= 0 if ε 2 A,
where 0(ξ) = 0 8ξ 2 Ξ. Figure 1 shows the fuzzy soft sets. The family of all these fuzzy soft sets over Ξ denoted by $ΦΣΣ(Ξ)_E$.
Proposition 1.1.1. [12] Every fuzzy set may be considered a soft set.
Obviously, a classical soft set $Φ_A$ over a universe X can be seen as a fuzzy soft set by using the characteristic function of the set $Φ_A(ε)$:
$Φ_A(ε)(α)$ = $_{Φ_A(ε)}(α)$ =
_ 1 if α 2 $Φ_A(ε)$;
0 otherwise
De.nition 1.1.12. [8,85] The complement of a fuzzy soft set $(ϕ_A;E)$, denoted by $(ϕ;A)_χ$ or $ϕ_{χA}$
, is de.ned by $(ϕ;A)_χ$ = $(ϕ_χ;A)$; $ϕ_{χA}$
: E ! $I_Ξ$ is a mapping given by
3
Figure 1: [17] shows a fuzzy soft set
$_^ε$
$_{ϕ_{χA}}$
= 1 − $_^ε$
$_{ϕ_A}$ 8ε 2 A where 1(ξ) = 1 8ξ 2 Ξ. Clearly, ($ϕ_{χA}$

$)^{\chi} = \phi_A$.

Definition 1.1.13. [85] A fuzzy soft set ϕ_A over Ξ is said to be a NULL fuzzy soft set, denoted by $\tilde{0}_A$, if for all $\varepsilon \in A$, $\phi_A(\varepsilon) = 0$.

Definition 1.1.14. A fuzzy soft set ϕ_A over Ξ is said to be an absolute fuzzy soft set, denoted by $\tilde{1}_A$, if for all $\varepsilon \in A$, $\phi_A(\varepsilon) = 1$. Clearly we have, $(\tilde{1}_A)^{\chi} = \tilde{0}_A$ and $(\tilde{0}_A)^{\chi} = \tilde{1}_A$.

Definition 1.1.15. [85] Let $\phi_A; \gamma_B \in \Phi\Sigma\Sigma(\Xi)_E$. Then, ϕ_A is fuzzy soft subset of γ_B, denoted by $\phi_A \tilde{\sqsubseteq} \gamma_B$, if $A \subseteq B$ and $\phi_A^{\varepsilon}(\xi) \leq \gamma_B^{\varepsilon}(\xi)$ $\forall \xi \in \Xi; \varepsilon \in A$. Also ϕ_A is called a fuzzy soft superset of γ_B if γ_B is a fuzzy soft subset of ϕ_A, and we write $\phi_A \tilde{\sqsupseteq} \gamma_B$.

Definition 1.1.16. [85] Let ϕ_A, $\gamma_B \in \Phi\Sigma\Sigma(\Xi)_E$. Then ϕ_A and γ_B are said to be equal, denoted by $\phi_A = \gamma_B$, if $\phi_A \tilde{\sqsubseteq} \gamma_B$ and $\phi_A \tilde{\sqsupseteq} \gamma_B$.

Definition 1.1.17. [8,21,85] The union of two fuzzy soft sets ϕ_A and γ_B over the

common universe Ξ is also a fuzzy soft set η_X, where $X = A \cup B$ and for all $\varepsilon \in X$, $\eta_X(\varepsilon) = \eta_X^{\varepsilon} = \phi_A^{\varepsilon} \vee \gamma_B^{\varepsilon}$ $\forall \varepsilon \in E$. Here we write $\eta_X = \phi_A \sqcup \gamma_B$.

Definition 1.1.18. [8, 21,85] The intersection of two fuzzy soft sets ϕ_A and γ_B over the common universe Ξ is also a fuzzy soft set η_X, where $X = A \cap B$ and for all $\varepsilon \in X$, $\eta_X(\varepsilon) = \eta_X^{\varepsilon} = \phi_A^{\varepsilon} \wedge \gamma_B^{\varepsilon}$ $\forall \varepsilon \in E$. Here we write $\eta_X = \phi_A \sqcap \gamma_B$.

Theorem 1.1.1. [17,25] Let ϑ be an index set and ϕ_A, γ_B, η_X, $(\phi_A)_\iota; (\gamma_B)_\iota \in \Phi\Sigma\Sigma(\Xi)_E$ $\forall \iota \in \vartheta$, then:

(1) $\phi_A \sqcap \phi_A = \phi_A$; $\phi_A \sqcup \phi_A = \phi_A$.

(2) $\phi_A \sqcap \gamma_B = \gamma_B \sqcap \phi_A$, $\phi_A \sqcup \gamma_B = \gamma_B \sqcup \phi_A$.

(3) $\phi_A \sqcup (\gamma_B \sqcup \eta_X) = (\phi_A \sqcup \gamma_B) \sqcup \eta_X$, $\phi_A \sqcap (\gamma_B \sqcap \eta_X) = (\phi_A \sqcap \gamma_B) \sqcap \eta_X$:

(4) $\phi_A \sqcap (\sqcup_{\iota \in \vartheta}$

(γB) ι) = t ι2ϑ
(ϕA u (γB) ι), ϕA t (u ι2ϑ
(γB) ι) = u ι2ϑ
(ϕA t (γB) ι) :
(5)e0
E e_
ϕA e_
e1
A e_
e1
E.
(6) [u ι2ϑ
(ϕA) ι]χ = t ι2ϑ
(ϕA) χι
, [t ι2ϑ
(ϕA) ι]χ = u ι2ϑ
(ϕA) χι
.
(7) If ϕA e_
γB, then γχB
e_
ϕχA
.
(8) ϕA u γB e_
ϕA, γB and ϕA, γB e_
ϕA t γB:
(9) ϕA e_
γB , ϕA = ϕA u γB and γB = ϕA t γB.
De.nition 1.1.19. [1, 98] The di¤erence of two fuzzy soft sets ϕA and γB over
the common universe Ξ , denoted by ϕA e
γB; is also a fuzzy soft set ηX, where
X = A\B 6= _ and 8 ε 2 X, 8ξ 2 Ξ; _ε
ηX (ξ) = minf_ε
ϕA (ξ) ; 1_ε
γB (ξ) g: Clearly, ϕA
e
γB = ϕA u γχB
.
De.nition 1.1.20. [25,97,110] Let _ be a collection of fuzzy soft sets over a universe
Ξ with a .xed set of parameters E, then _ _ ΦΣΣ (Ξ) E is called fuzzy soft topology
on Ξ if:
1-e0

E,e1
E 2 _ wheree0
E(ε) = 0 ande1
E(ε) = 1 8ε 2 E,
2-the union of any members of _ belongs to _ ,
3-the intersection of any two members of _ belongs to _ .
The triplet (Ξ; _ ;E) is called fuzzy soft topological space over Ξ. Also, each
5
member of _ is called fuzzy soft open in (Ξ; _ ;E). We denote the set of all fuzzy soft open sets by ΦΣΟ(Ξ)E.
Example 1.1.1. [118]
_ 0 = fe0
E;e1
Eg is a fuzzy soft indiscrete topology on Ξ.
_ 1 = ΦΣΣ(Ξ)E is a fuzzy soft discrete topology on Ξ.
Note that, the intersection of any family of fuzzy soft topologies on Ξ is also a fuzzy soft topology on Ξ [97].
De.nition 1.1.21. [110] Let (Ξ; _ ;E) be a fuzzy soft topological space. A fuzzy soft set φA over Ξ is said to be fuzzy soft closed set in Ξ, if its relative complement
φχA
is fuzzy soft open set.
Theorem 1.1.2. [16, 77] Let (Ξ; _ ;E) be a fuzzy soft topological space and let _ χ denote the collection of all fuzzy soft closed sets. Then:
(1)e0
E,e1
E 2 _ χ.
(2) If φA, γB 2 _ χ, then φA t γB 2 _ χ.
(3) If (φA)ι 2 _ χ, 8ι 2 ϑ, then u ι2ϑ
(φA)ι 2 _ χ.
De.nition 1.1.22. [82] A fuzzy soft topology _ 1 is called weaker (or coarser) than a fuzzy soft topology _ 2 if and only if _ 1 _ _ 2 . In that case _ 2 is said to be stronger (or .ner) than _ 1.
De.nition 1.1.23. [82] Let (Ξ; _ ;E) be a fuzzy soft topological space and Ψ _ Ξ.
Let ηΨ
E be a fuzzy soft set over (Ψ;E) such that ηΨ
E : E □! IΨ such that ηΨ
E(ε) = _ε
ηΨ
E
;
_ε
ηΨ
E
(ξ) =
_ 1 ; ξ 2 Ψ

0 ; ξ =2 Ψ
Let _ Ψ = fηΨ
E u γB : γB 2 _g then the fuzzy soft topology _ Ψ on (Ψ;E) is called
fuzzy soft subspace topology for (Ψ;E) and (Ψ; _ Ψ ;E) is called fuzzy soft subspace of
(Ξ; _ ;E). If ηΨ
E 2 _ (respectively, ηΨ
E 2 _ χ), then (Ψ; _ Ψ ;E) is called fuzzy soft open
subspace (respectively, fuzzy soft closed subspace) of (Ξ; _ ;E).
De.nition 1.1.24. [82] Let (Ξ; _ ;E) be a fuzzy soft topological space and γB be a
6
fuzzy soft subset of Ξ. Then _ γB = fγB u ϕA; ϕA 2 _g is called fuzzy soft relative
topology and (γB; _ γB;B) is called fuzzy soft subspace. If γB 2 _ , then (γB; _ γB;B)
is called fuzzy soft open subspace. If γB 2 _ χ, then (γB; _ γB;B) is called fuzzy soft
closed subspace.
De.nition 1.1.25. [118] Let (Ξ; _ ;E) be a fuzzy soft topological space and a be a
subfamily of _ . If every element of _ can be written as an arbitrary fuzzy soft union
of some elements of a, then a is called a fuzzy soft basis for the fuzzy soft topology
_ .
Theorem 1.1.3. [118] A subfamily a of _ is called a fuzzy soft open base or simply
a base of fuzzy soft topological space (Ξ; _ ;E) if the following conditions hold:
(1)e1
E can be expresed as a fuzzy soft nuion of elements of a;
(2) If ϕA, γB 2 a; then 9ηX 2 a such that ηX e_
ϕA u γB .
Theorem 1.1.4. [105] Let _ be a fuzzy soft base for a fuzzy soft topology _ _ on Ξ.
Then, ϕA 2 _ _ if and only if ϕA = t _2_
(γB)_ where (γB)_ 2 _ for each _ 2 _.
De.nition 1.1.26. [118] Let (Ξ; _ ;E) be a fuzzy soft topological space. A sub-
collection _ of _ is called a subbase for _ if the family of all .nite intersections of
members of _ forms a base for _ .
Theorem 1.1.5. [118]Let _ be a family of fuzzy soft sets over Ξ such that e1
E, e0
E
2 _. Then, _ is a base for the topology _ , whose members are of the form t ι2ϑ
(u κ2_ι
(ϕA)ι;κ), where ϑ is arbitrary index set and for each ι 2 ϑ, _ι is a .nite index set,
(ϕA)ι;κ 2 _ for ι 2 ϑ and κ 2 _ι.
De.nition 1.1.27. [67] The property Π is said to be a hereditary property if
(Ξ; _ ;E) is a fuzzy soft topological space has the property Π, then every fuzy soft
subspace has the Π.
7
1.2 Some fuzzy soft topological concepts
De.nition 1.2.1. [83] The fuzzy soft set ϕA 2 ΦΣΣ(Ξ)E is called fuzzy soft point

if there exist ξ 2 Ξ and ε 2 E such that
ϕA (α) =
8><
>:
ξ_ α = ε
0 α 2 E fεg
; 0 < _ _ 1
This fuzzy soft point is denoted by ξε
_ or ϕε. The class of all fuzzy soft points of
Ξ is denoted by ΦΣΠ (Ξ) E.
De.nition 1.2.2. [82] Two fuzzy soft points are said to be disjoint, denonted by
ξε
_ 6= ψτ_
, if ξ 6= ψ; or _ 6= _; or ε 6= τ:
De.nition 1.2.3. [83] The fuzzy soft point ξε
_ is said to be belonging to the fuzzy
soft set ϕA, denoted by ξε
_ e2
ϕA, if for the element ε 2 A, _ _ _ε
ϕA (ξ) .
Theorem 1.2.1. [83] Let (Ξ; _ ;E) be a fuzzy soft topological space and ϕε be a
fuzzy soft point. Then the following properties hold:
1- If ϕε e2
γB, then ϕε
e=
2 γχB
,
2- ϕε e2
γB ; ϕχ
ε e2
γχB
;
3-Every non-null fuzzy soft set ϕA can be expressed as the union of all the fuzzy soft
points belonging to ϕA.
De.nition 1.2.4. [54] A fuzzy soft set γB in a fuzzy soft topological space (Ξ; _ ;E)
is called fuzzy soft neighborhood of the fuzzy soft point ξε
_ if there exists a fuzzy soft
open set ηX such that ξε
_ e2ηX e_
γB. A fuzzy soft set γB in a fuzzy soft topological
space (Ξ; _ ;E) is called fuzzy soft neighborhood of the fuzzy soft set ϕA if there
exists a fuzzy open soft set ηX such that ϕA e_
ηX e_
γB. The fuzzy soft neighborhood