



**Ain Shams University
Faculty of Education
Mathematics Department**

Codes and Some Results Based on Algebraic Structures

A thesis submitted to mathematics department
Faculty of Education - Ain Shams University
In partial fulfillment of the requirements for the
Master's degree in teacher preparation in science
(Pure Mathematics)

By

Hussein Aly Hussein Jad

Supervised by

**Dr. Samy Mohamed Mostafa
Ass. Prof. of pure Mathematics
Faculty of Education
Ain Shams University**

**Prof. Dr. Bayumy Youssef
Professor of Computer graphics
Informatics Research Institute
City for Scientific Research and
Technology Applications**

(2016)



Ain shams University
Faculty of Education
Mathematics Department

Candidate: Hussein Aly Hussein Jad.

Thesis title: Codes and Some Results Based on Algebraic Structure

Degree: Master's degree in teacher preparation in sciences.
(Pure Mathematics)

Supervisors:

Signature

Dr. Samy M. Mostafa
Ass. Prof. of Mathematics
Faculty of Education
Ain Shams University

Prof. Dr. Bayumy Youssef
Professor of Computer graphics
Informatics Research Institute
City for Scientific Research and Technology Applications

(2016)

Acknowledgements

First and foremost, I deeply thank Allah for his virtues and graces on me.

Secondly, I would like to thank my family for their infinite support and help; I dedicate all my achievements to them.

My sincere thanks to my supervisors from which I learned a lot of things.....

Great thanks to **Dr. Samy Mostafa**, associate professor, at the Department of Mathematics, Faculty of education, Ain shams University for his valuable advices, kind guidance and cooperation through the work. I acquire from him the knowledge to enhance my research and serve my career.

Special thanks to **Prof .Dr. Bayumy Youssef**, professor at the Department of Computer Graphics, Informatics Research Institute, City for Scientific Research and Technology Applications for his cooperation and motivation during my research work and thesis writing. I really enjoyed working with him.

My deep thanks to **Prof .Dr. Osman Mohamed** .the head of Mathematics department, Faculty of education, Ain shams University for his interest and help through this thesis.

I want to express my deepest gratitude to my friends for their helping and encouragement.

Contents

Summary	I
---------	---

Chapter one: Basic concepts and Related Work

1.1 BCK - Algebras	1
1.2 KU-algebras	6
1.3 PU-Algebras	9
1.4 R0 -Algebras	14
1.5 Hilbert Algebras	17
1.6 Some Concepts of Graph	19
1.7 Binary Block Codes	27
1.8 Soft sets	36

Chapter Two: Coding Theory Applied to KU-Algebras

2.1 KU-Valued Functions	42
2.2 Codes Generated by KU-Functions	47

Chapter Three: Efficient Algorithm for Constructing KU-Algebras

from Block Codes

3.1 Basic Results	50
3.2 Relationship between the Ordered Relation on KU-Algebra and Partial Ordered Set	57

Chapter Four: Intersectional (α, A) - Soft New-Ideals in PU-Algebras

4.1 Soft PU-Algebras	63
4.2 Intersectional (α, A) – Soft New PU- Ideals	65

Reference	69
-----------	----

Appendix	75
----------	----

PREFACE

The notion of BCK-algebras was proposed by Iseki and Iami ([25], [31]) in 1966. Iseki [32] introduced the notion of a BCI-algebra, which is a generalization of BCK-algebra. Since the numerous mathematical papers have been written to investigate the algebraic properties of the BCK/BCI-algebras and their relationship with other structures ([13], [14], [15]). So, there is a great deal of literature which has been produced on the theory of BCK/BCI-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role.

Prabpayak and Leerawat ([57], [58]) introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. Also, they introduced the concept of homomorphism of KU-algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. These algebras form an important class of logical algebras and have many applications to various domains of mathematics, such as, group theory, functional analysis, fuzzy sets theory, probability theory, topology, etc. BCK-algebras also form an important class of logical algebras introduced by Iseki ([28], [31], [32]) and were extensively investigated by several researchers. Iseki posed an interesting problem (solved by Wronski [68]) whether the class of BCK-algebras is a variety.

Coding theory is a very young mathematical topic. It started on the basis of transferring information from one place to another. For instance, suppose we are using electronic devices to transfer information (telephone, television, etc.). Here, information is converted into bits of 1's and 0's and sent through a channel, for example a cable or via satellite. Afterwards, the 1's and 0's are reconverted into information again. The idea of coding theory is to present a method of how to convert the information into bits, such that there are no mistakes in the received information, or such that at least some of them are corrected. One of the recent applications of BCK-algebras was presented in the Coding theory ([18], [33]). Jun et al [33] introduced the notion of BCK-valued functions and

investigate several properties. Also, they established block-codes by using the notion of BCK-valued functions and presented that every finite BCK-algebra determines a block-code. Flaut [18] provided an algorithm which allows to find a BCK-algebra starting from a given binary block code. Saeid et al [60] presented some new connections between BCK- algebras and binary block codes. Over the last 70 years, algebraic coding has become one of the most important and widely applied aspects of abstract algebra. Coding theory forms the basis of all modern communication systems, and the key to another area of study that is Information Theory. Coding theory is the study of methods for efficient and accurate transfer of information from one party to another. Various types of codes and their connections with other mathematical objects have been intensively studied.

Mostafa et al [44] introduced a new algebraic structure called PU-algebra, and they investigated several basic properties. Moreover, they derived new view of several ideals on PU-algebra. The concept of fuzzy sets was introduced by Zadeh [72]. In 1991, Xi [70] applied the concept of fuzzy sets to BCI, BCK, MV -algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has been developed in many directions and applied to a wide variety of fields. Mostafa et al [45] introduced the notion of α -fuzzy and $(\tilde{\alpha}, \alpha)$ -cubic new-ideal of PU -algebra. They discussed the holomorphic image (pre image) of α -fuzzy and $(\tilde{\alpha}, \alpha)$ -cubic new-ideal of PU -algebra. Molodtsov [54] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Maji et al ([49], [50], [51]) described the application of soft theory and studied several operations on the soft sets. Many Mathematicians have studied the concept of soft set of some algebraic structures. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çagman et al ([4], [10], [11]) introduced fuzzy parameterized (FP) soft sets and their related properties. Jun [33] applied Molotov's notion of soft sets to the theory of BCK/BCI-algebras. Jun and Park [35] deal with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras and gave several examples. Jun et al [36] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and investigated their basic properties. Moreover, Jun et al [37] Applied a fuzzy soft set introduced by Maji et al [29] as a generalization of the

standard soft sets for dealing with several kinds of theories in BCK/BCI-algebras. They defined the notions of fuzzy soft BCK/BCI-algebras, fuzzy soft ideals, and fuzzy soft p-ideals, and investigated related properties. Yang et al [69] introduced the concept of the interval-valued fuzzy soft set. Also, they studied the algebraic properties of the concept.

Aim of the thesis:

The aim of this thesis is to introduce codes of a KU-algebras and present algorithms for constructing codes from KU-algebras and conversely. Moreover the notions of intersectional A -soft new-ideals and intersectional (α, A) -soft new-ideals in PU-algebras are introduced.

This thesis is broadly divided into four chapters

Chapter one:

Highlights on some basic definitions and related work results available in the standard literature.

Chapter two:

The notion of KU-valued functions is introduced and several properties are investigated. Moreover, block codes by using the notion of KU-valued functions are established and some new connections between KU- algebras and binary block codes are presented.

Chapter three:

Provides an algorithm which allows to find a KU-algebras starting from a given binary block code.

Chapter four:

The new notions, intersectional A - soft new-ideals, intersectional (α, A) -soft new-ideals in PU-algebras and their properties are investigated. Also, the relations between an intersectional A - soft new-ideals and an intersectional (α, A) -soft new-ideals are provided. Moreover, the homomorphic image of an intersectional (α, A) -soft new-ideals is studied.

LIST OF PUBLICATIONS

- 1- Mostafa, S. M., Youssef, B., & Jad, H. A. (2015). Coding Theory Applied To KU-Algebras. Journal of New Theory, 6, 43-53.
- 2- Mostafa, S. M., Youssef, B., & Jad, H. A. (2016). Efficient Algorithm for Constructing KU-algebras from Block Codes. International Journal of Engineering and Science Invention, 5, may2016.
- 3- Mostafa, S. M., Kareem, F. F., & Jad, H. A. (2016). Intersectional (α, A) - Soft New-Ideals in PU-Algebras. Journal of New Theory, 13, may2016.

Chapter One

Basic Concepts and Related Work

Chapter One

Basic Concepts and Related Work

In this chapter, all necessary concepts and related work which will be used in our thesis and appropriate for others to have are presented. The proofs of most results are well-known and we just state what we need in our thesis without proofs.

§ (1.1) *BCK*-Algebras

BCK-algebra is an important class of logical algebras that was first introduced in mathematics in 1966 by Y. Imai and K. Iséki. These algebra have many applications to various domains of mathematics. In this section, we state the basic results that referred to ([13], [24], [25], [26], [27], [29], [30], [39], [41], [61], [64], [65]).

Definition 1.1.1. Let X be a non-empty set with a binary operation "*" and a constant 0. The system $(X, *, 0)$ is called a *BCI*- algebra if it satisfies the following axioms for all $x, y, z \in X$.

$$(BCI-1) \quad ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCI-2) \quad (x * (x * y)) * y = 0,$$

$$(BCI-3) \quad x * x = 0,$$

$$(BCI-4) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

If a *BCI*-algebra X satisfies the identity $0 * x = 0$, for all $x \in X$, then X is called a *BCK* -algebra. It is show that the class of *BCK* -algebras is a proper subclass of the class of *BCI* -algebras. For brevity we call X as a *BCK* - algebra.

In X we can define a binary relation $\leq$ by: $x \leq y$ if and only if $x * y = 0$. then the system $(X, *, 0)$ is a *BCK*- algebra if and only if it satisfies that:

$$(BCI'-1) \quad ((x * y) * (x * z)) \leq (z * y),$$

$$(BCI'-2) \quad (x * (x * y)) \leq y,$$

$$(BCI'-3) \quad x \leq x,$$

$$(BCI'-4) \quad x \leq y \text{ and } y \leq x \text{ imply } x = y,$$

$$(BCI'-5) \quad 0 \leq x,$$

$$(BCI'-6) \quad x \leq y \text{ if and only if } x * y = 0.$$

Example 1.1.2. Let $X = \{0,1,2\}$ in which $*$ defined by the following table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

It is clear that X is BCK - algebra.

In a BCK - algebra $(X, *, 0)$, the following properties are satisfied:

- 1) $x \leq y$ implies $z * x \leq z * y$,
- 2) $x \leq y$ and $y \leq z$ implies $x \leq z$,
- 3) $(x * y) * z = (x * z) * y$,
- 4) $(x * y) \leq z$ implies $(x * z) \leq y$,
- 5) $(x * z) * (y * z) \leq (x * y)$,
- 6) $x \leq y$ implies $x * z \leq y * z$,
- 7) $(x * (x * y)) * (y * x) \leq x * (x * (y * (y * x)))$,
- 8) $(x * y) \leq x$,
- 9) $x * 0 = x$.

Definition 1.1.3. Let $(X, *, 0)$ be a *BCK* – algebra, and let S be a non-empty subset of X , then S is called a sub-algebra of X , if $\forall x, y \in S, x * y \in S$.

Theorem 1.1.4. Suppose that $(X, *, 0)$ is a *BCK* - algebra and let S be a sub-algebra of X then:

- 1) $0 \in S$,
- 2) $(S, *, 0)$ is also a *BCK* – algebra,
- 3) X is a sub-algebra of X ,
- 4) $\{0\}$ is also a sub-algebra of X .

Example 1.1.5. Let $X = \{0, 1, 2\}$ in which $*$ is defined by the following table:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $S = \{0, 2\}$ is *BCK* sub-algebra of X .

Lemma 1.1.6. In a *BCK* – algebra $(X, *, 0)$, for all $x, y, z \in X$:

- 1) if $x \neq y$ then $y * x \neq 0$, whenever $x * y = 0$,
- 2) $x * y = z$ implies $z * x = 0$.

Definition 1.1.7. A *BCK*–algebra $(X, *, 0)$ is said to be commutative if it satisfy

$$\forall x, y \in X, x * (x * y) = y * (y * x). \text{ i.e. } x \wedge y = y \wedge x.$$

Theorem 1.1.8. A *BCK*–algebra X is commutative if and only if $x * (x * y) = y * (y * (x * (x * y)))$, for all $x, y \in X$.

Theorem 1.1.9. For a *BCK*–algebra X , the following conditions are equivalent:

- 1) X is commutative ,
- 2) $x * (x * y) \leq y * (y * x)$,
- 3) $(x * (x * y)) * (y * (y * x)) = 0$.

Example 1.1.10. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	0
3	3	3	2	0

Then $(X, *, 0)$ is a commutative *BCK*-algebra.

Definition 1.1.11. A *BCK*–algebra X is called to be positive implicative if it satisfies

$$\forall x, y, z \in X, (x * z) * (y * z) = (x * y) * z.$$

Example 1.1.12. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a positive implicative *BCK* -algebra.

Theorem 1.1.13. An Algebra $(X, *, 0)$ of type $(2, 0)$ is positive implicative *BCK*-algebra if and only if it satisfies *BCI* -1, *BCK* -5 and

- 1) $(x * (x * y)) * (y * x) = (y * (y * x)) * (x * y)$,
- 2) $x * 0 = x$.

Theorem 1.1.14. Let $(X, *, 0)$ be a *BCK* -algebra then the following conditions are equivalent:

- 1) X is a positive implicative ,
- 2) $(x * y) * z = 0$ implies $(x * z) * (y * z) = 0$,
- 3) $(x * y) * y = 0$ implies $x * y = 0$.

Theorem 1.1.15. Let $(X, *, 0)$ be a *BCK* – algebra then the following conditions are equivalent:

- 1) X is a positive implicative ,
- 2) $x * y = (x * y) * y$,
- 3) $(x * (x * y)) * (y * x) = x * (x * (y * (y * x)))$,
- 4) $x * y = (x * y) * (x * (x * y))$,
- 5) $x * (x * y) = (x * (x * y)) * (x * y)$,
- 6) $(x * (x * y)) * (y * x) = (y * (y * x)) * (x * y)$.

Definition 1.1.16. A *BCK*–algebra $(X, *, 0)$ is said to be implicative if it satisfy

$$\forall x, y \in X, x = x * (y * x).$$

Theorem 1.1.17. A *BCK*–algebra is implicative if and only if it is both commutative and positive implicative.

Example 1.1.18. Let $X = \{0, 1, 2\}$ in which $*$ is defined by the following table:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is implicative *BCK* -algebra.

Definition 1.1.19. A non-empty subset I of a BCK -algebra X is called BCK ideal of X if it satisfies the following conditions:

- 1) $0 \in I$,
- 2) $x * y \in I, y \in I$ Implies $x \in I, \forall x, y \in X$.

Theorem 1.1.20. Suppose I is an ideal of a BCK -algebra X and $x \in I$, if $y \leq x$ then $y \in I$.

Example 1.1.21. Let $X = \{0, 1, 2\}$ in which $*$ is defined by the following table:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is an implicative BCK -algebra, and $\{0\}, X, \{0, 1\}, \{0, 2\}$ are all ideals of X .

§ (1.2) KU -Algebras

C. Prabpayak and U. Leerawat ([57], [58]) introduced a new algebraic structure which is called KU -algebras and investigated some related properties. In this section, we present the basic definitions and results which are related to KU -algebras. The main results in this section are taken from ([47], [48]).

Definition 1.2.1. Let X be a set with a binary operation " $*$ " and a constant 0 . $(X, *, 0)$

is called KU -algebra if the following axioms hold $\forall x, y, z \in X$:

$$KU_1 \quad (x * y) * ((y * z) * (x * z)) = 0,$$

$$KU_2 \quad x * 0 = 0,$$

$$KU_3 \quad 0 * x = x,$$