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# EXACT SOLUTIONS FOR SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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# SUMMARY

The objective of this thesis is to study the exact solutions for some nonlinear partial differential equations, and some equations with variable coefficients in order to find new exact solutions by applying various techniques. These techniques are: symmetry, first integral, direct integral, modified extended tanh- function, modified Kudryashov, homotopy perturbation and variational iteration methods. In addition, we present a new method which is called new extended rational- function method.

This thesis consists of five chapters, together with an Arabic and English summaries. It is organized as follows:

**Chapter (I)**: Comprise an introduction contained in a brief survey. Development of the available literature relevant to the work given in chapters II-V and includes the general notations and mathematical tools.

**Chapter (II)**: Here in Calogero-Bogoyvlenskii-Schiff equation and the coupled Burgers-type equations which appear in a wide variety of physical applications, have been analyzed via symmetry method. Also, using the infinitesimal symmetries, there are six basic fields determined for the first equation while in the second only two basic fields are obtained. These fields which help us to reduce the first equation into partial differential equations with two variables and the second system also reduced to nonlinear system of ordinary differential equations. For each case of the six cases arises in the first equation, the reduced partial differential equations are transformed to nonlinear ordinary differential equations. The search for the solutions of those reduced ordinary equations, corresponding to the equation under consideration, has yielded new certain classes of exact solutions for the Calogero-Bogoyvlenskii-Schiff equation and the coupled Burgers-type equations.

**Chapter (III)**: The objective of this chapter is to study the exact solutions of the nonlinear partial differential equations with variable coefficients in order to find new exact solutions by applying the first integral and the direct integral methods.

This chapter contains eight sections. In the first and second sections, we have studied the Long-Short wave resonance equations. In the third and fourth sections, we study the nonlinear Schrödinger equation with variable coefficients. In the fifth and sixth sections, we study the two-dimensional Burger equation with variable coefficients. In the seventh and eighth sections, we study the (2+1)-dimensional Broer-Kaup system with variable coefficients. The application of the first integral and the direct integral methods yields many exact solutions in the form of trigonometric, hyperbolic and Jacobi elliptic functions.

**In Chapter (IV)**, we make use of the modified extended tanh-function and new extended rational-function methods for finding an exact solution of Zabolotskay-Khoklov equation (Burger's equation in two-space dimension). Also the modified Kudryashov method with the aid of symbolic computation has been applied to obtain exact solutions of the (2+1)-dimensional modified Korteweg-de Vries equation and nonlinear Drinfeld-Sokolov system.

The third and fourth sections have been published in American Journal of Computational and Applied Mathematics.

**In Chapter (V)**, we study Zabolotskay-Khoklov equation (Burger's equation in two-space dimension), and the (2 + 1)-dimensional Konopelchenko-Dubrovsky (KD) equations by using variational iteration (VIM) and homotopy perturbation methods (HPM). Comparison between the exact solutions obtained before and solutions obtained by HPM and VIM has been made to judge how accurate solutions are those methods.

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# CHAPTER I

# INTRODUCTION, SURVEY OF LITERATURE AND MATHEMATICAL TOOLS

## *1.1 Introduction*

In many different fields of science and engineering, it is very important to obtain exact solutions for nonlinear partial differential equations. In recent years, the investigation of those exact solutions plays an important role in the study of nonlinear physical phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics, hydrodynamic, nonlinear optic, and many more see for example [1,2,3,9,20,32,35,37,39,47,48,64,66,70].

Now, we proceed our original task of surveying the mathematical tools utilized in this thesis. The first method is the symmetry method given by Steinberg [52]. In his report he discussed a method of finding explicit solutions of both linear and nonlinear partial differential equations (PDEs). The classical version of his method is usually referred to as similarity method.

Steinberg's method is more algorithmic and more general than the classical Lie method [11, 12, 43]. In the case of linear equations this method bear a close relationship to the method of separation of variables and, in fact, this method produces a large supply of separable solutions for linear equations.

One of the most important thing for Steinberg's symmetry method is that it is a computational procedure that can be used by any person familiar with differential equations but not familiar with Lie group theory [11, 12, 43] . Although the person who is familiar with Lie group theory will find that the elementary methods that used group theory to find similarity solutions are not powerful enough to give a complete analysis of the heat and Burger equations as the symmetry method of Steinberg gave in his report. Bhutani et al in [8] enlarged this technique to deal with systems of partial differential equations then the technique has found an important place in the literature of group theoretic methods see $([19], [39-41])$.

The second method is the modified extended tanh-function method. It is first proposed by Fan [21] to obtain new travelling wave solutions that can not be obtained by tanh-function method. Most recently, El-Wakil et. al. [20] presented the modified extended tanh-function method to obtain more new exact solutions.

The third method is the first-integral method. It is first proposed by Feng [22] to solve the Burgers-KdV equation. This method based on the ring theory of commutative algebra. Also, this useful method has been widely used by many researchers, such as in [1, 46] and the references therein.

The fourth method is the modified Kudryashov method. It is proposed by [32, 61]. The matter of this method is the modification of the approach by Kudryashov to get new rational solution in the Exp-function.

The fifth method we present is the new extended rational-function method for constructing new exact solutions of nonlinear partial differential equations (NPDEs).

The sixth method is the homotopy perturbation method (HPM). It is first proposed by He [27] for solving linear and nonlinear differential and integral equations. This method has been the subject of extensive analytical and numerical studies. Also, it has the significant advantage that it provides an approximate solution to a wide range of nonlinear problem in applied science. The method is a coupling of the traditional perturbation method and homotopy in topology. In topology, a homotopy is constructed with an embedding parameter. In order to know more applications of this method, we refer the interested readers to [9, 27, 28] and references therein.

The seventh method is the variational iteration method (VIM), which proposed by He [29], is effectively and easily used to solve some classes of nonlinear problems. In order to know more applications of this method, we refer the interested reader to [4, 45] and references therein. Moreover, in this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers in the functionals can be identified optimally via the variational theory. The initial approximations can be freely chosen with possible unknown constants, which can be determined by imposing the boundary/initial conditions. So that, the sixth and seventh methods do not require linearization or small perturbation.

After giving a brief survey of the available literature relevant to the work put up in chapters II-V, we reproduce in the next section, necessary definitions, general notations, and mathematical tools essential to understand and carry over the methods utilized.

## *1.2 Symmetry method*

We briefly outlined Steinberg's [52] similarity method of finding explicit solutions of both linear and nonlinear partial differential equations . The method based on finding the symmetries of the differential equation as follows:

Suppose that the differential operator $L$ can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u) \tag{1.1}$$

where $u = u(t, x)$ and $H$ may depend on $t, x, u$ and any derivative of $u$ as long the derivative of $u$ dose not contain more than $p-1$ , $t$ derivatives. Consider the symmetry operator (called infinitesimal symmetry) which is quasi-linear partial differential operator of first-order, has the form

$$S(u) = A(t, x, u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t, x, u)\frac{\partial u}{\partial x_i} + C(t, x, u). \tag{1.2}$$

Define the Frèchet derivative of $L(u)$ by

$$F(L,u,v)=\frac{d}{d\varepsilon}L(u+\varepsilon v)|_{\varepsilon=0}. \tag{1.3}$$

With these definitions in mind we need to follow the following steps:

(i) Compute $F(L,u,v)$.

(ii) Compute $F(L,u,S(u))$.

(iii) Substitute $H(u)$ for $(\frac{\partial^p u}{\partial t^p})$ in $F(L,u,S(u))$.

(iv) Set this expression to zero and perform a polynomial expansion.

(v) Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of $S(u)$, equation under study can be used to obtain the functional form of the solutions.

In the case of linear and nonlinear partial differential system, we use the following two symmetry operators

$$\begin{aligned} S_1(u,v) &= A(x,t,u,v)u_t+B(x,t,u,v)u_x+C_1(x,t,u,v), \\ S_2(u,v) &= A(x,t,u,v)v_t+B(x,t,u,v)v_x+C_2(x,t,u,v). \end{aligned} \tag{1.4}$$

and by the same way described above we calculate Frèchet derivatives $F_1$and $F_2$ which are corresponding to the symmetry operators $S_1(u,v)$ and $S_2(u,v)$, thereby we use them to obtain the functional form of the solution.

## 1.3 Modified extended tanh-function method

To illustrate the basic concepts of the modified extended tanh-function method as given in $[5,20,21]$ , consider a given PDE in two independent variables given by

$$F\left(u,u_t,u_x,u_{xx},...\right)=0. \tag{1.5}$$

We first consider its travelling solution $u = U(\theta)$, $\theta = x + \lambda t$ or $\theta = x - \lambda t$ then Eq. (1.5) becomes an ordinary differential equation as follows

$$F(U, U', U'', ...) = 0. \tag{1.6}$$

Then, we propose the following series expansion as a solution of (1.6):

$$u(x,t) = U(\theta) = \sum_{i=0}^{m} \left( \phi^{i}(\theta) A_i + \phi^{-i}(\theta) B_i \right), \tag{1.7}$$

where $\phi = \phi(\theta)$ satisfies the following Riccati equation

$$\phi'(\theta) = b + \phi^2(\theta), \tag{1.8}$$

where $b$ is a constant to be determined, $\phi'(\theta) = \frac{d\phi}{d\theta}$. The positive integer parameter $m$ can be determined by balancing the highest derivative term with nonlinear term in (1.6). Substituting (1.7) and (1.8) into (1.6) and then equating to zero all coefficients of $\phi^i$, we can obtain a system of algebraic equations, from which the constants $b, \lambda, A_0, A_1, ..., A_m, B_1, ..., B_m$ are obtained explicitly. Fortunately, Riccati admits several types of solutions

$$\begin{aligned}
\phi(\theta) &= -\sqrt{-b}\tanh(\sqrt{-b}\theta), && b < 0,\\
\phi(\theta) &= -\sqrt{-b}\coth(\sqrt{-b}\theta), && b < 0,\\
\phi(\theta) &= \sqrt{b}\tan(\sqrt{b}\theta), && b > 0,\\
\phi(\theta) &= -\sqrt{b}\cot(\sqrt{b}\theta), && b > 0,\\
\phi(\theta) &= -\frac{1}{\theta}, && b = 0.
\end{aligned} \tag{1.9}$$

The algorithm presented here is a computerizable method, in which generating an algebraic system for (1.6) can be easily solved by symbolic computation software like Mathematica or Maple programs.

## 1.4 First integral method

Consider the nonlinear partial differential equation in the from [22, 46]

$$F(S, S_x, S_t, S_{xx}, S_{tt}, S_{xt}, ...) = 0, \tag{1.10}$$

where $S = S(x,t)$ is a solution of the nonlinear partial differential equation (1.10). We use the transformation

$$S(x,t) = f(\xi), \tag{1.11}$$

where $\xi = x + \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = \lambda \frac{d}{d\xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \quad .... \tag{1.12}$$

By using Eq. (1.12), we can transfer the nonlinear partial differential equation (1.10) to a nonlinear ordinary differential equation of the form

$$G\left(f(\xi), \frac{df(\xi)}{d\xi}, \frac{d^2 f(\xi)}{d\xi^2}, ....\right) = 0. \tag{1.13}$$

Next, we introduce new independent variables

$$X(\xi) = f(\xi), \quad Y(\xi) = \frac{df(\xi)}{d\xi}, \tag{1.14}$$

which leads to system of nonlinear ordinary differential equations

$$\begin{aligned} \frac{dX(\xi)}{d\xi} &= \dot{X}(\xi) = Y(\xi), \\ \frac{dY(\xi)}{d\xi} &= \dot{Y}(\xi) = F_1(X(\xi), Y(\xi)). \end{aligned} \tag{1.15}$$

By the qualitative theory of ordinary differential equations [17] , if we can find the integrals of Eqs. (1.15) under the same conditions, then the general solutions of Eqs. (1.15) can be solved directly. However, in general, it is really difficult to realize this even for the first integral, because for a given plane autonomous system, there is no systematic theory that can tells us how to find its first integrals, nor logical way that tells us what these first integrals are?. We will apply the Division Theorem to obtain the first integral of Eqs. (1.15) which reduces Eq. (1.13) to a first order integrable ordinary differential equation. An exact solution of Eq. (1.10) is then obtained by solving this equation. Now, let us recall the Division Theorem:

**Division Theorem:** Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $C[\omega, z]$; and $P(\omega, z)$ is irreducible in $C[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $C[\omega, z]$ such that

$$Q(\omega, z) = P(\omega, z)\, G(\omega, z)$$

## 1.5 Modified Kudryashov method

To illustrate the basic idea of the modified Kudryashov method, we first consider a general form of nonlinear equation [32, 61]

$$p(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, ....) = 0, \tag{1.16}$$

where $p$ is a polynomial function with respect to the indicated variables or some function that can be reduced to a polynomial function by using some transformation.

Making use of the travelling wave transformation

$$u = u(\zeta), \quad \zeta = \alpha(x - \beta t), \tag{1.17}$$

where $\alpha$ and $\beta$ are arbitrary constants to be determined. Then Eq. (1.16) reduces to a nonlinear ordinary differential equation (ODE)

$$p(u, -\alpha\beta u', \alpha u', \alpha^2 u'', \alpha^2\beta^2 u'', -\alpha^2\beta u'', ....) = 0. \tag{1.18}$$

In this method, we shall seek a rational function type of solution for Eq. (1.18), in terms of $exp(\zeta)$, as follows

$$u(\zeta) = \sum_{k=0}^{m} \frac{\alpha_k}{[1 + \exp(\zeta)]^k}, \tag{1.19}$$

where $a_0, a_1,$ ......and $a_m$ are constants to be determined.

We can determine $m$ by balancing the linear term of the highest order in (1.18) with the highest order nonlinear term. Differentiating (1.19) with respect to $\zeta$, introducing the result into Eq. (1.18) and setting the coefficients of the same power of $e^{\zeta}$ equal to zero, we obtain a system of algebraic equations. The rational function solution of Eq. (1.16) can be solved by obtaining $a_0, a_1,$ ......,$a_m$ from this system.