Ain Shams University

Faculty of Science

Mathematics Department

Fixed and coincidence points for multi and single valued functions

A THESIS

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By

Salwa Hamdy Mohamed Mohamed

Under the supervision

Prof. Dr.Nashat Faried M. Fathi
Professor of Mathematics,
Faculty of Science,
Ain Shams University.

Dr.Fayza Abd-el Halem
Associate Professor of Mathematics,
Faculty of Science,
Ain Shams University.

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بسم الله الرحمن الرحيم

قال تعالى: "و قل اعملو فسيرى الله عملكم ورسوله والمؤمنون"

صدق الله العظيم

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Salwa Hamdy Mohamed

To my family

To my husband

To my friends

To everyone teach me

Contents

Introduction

The fixed point theorem, generally known as the Banach contraction principle, appeared in explicit form in Banach's thesis in ١٩٢٢ where it was used to establish the existence of the solution for an integral equation and it guarantees the existence and uniqueness of **fixed points** of certain self maps of metric spaces, and provides a constructive method to find those fixed points. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis (differential, integral, partial differential equations, and variational inequalities, etc.).

The fixed point problem (at the basis of fixed point theory) stated as:

A topological space X is said to have the **fixed point property** (briefly FPP) if for any continuous function there exists $x \in X$ such that f(x) = x and x is called a **fixed point** of f.

The formal definition of Banach spaces is due to Banach himself. But examples like the finite dimensional vector space $\mathbb{R}^n$ were studied prior to Banach spaces.

Let (X, d) be a metric space. Then a mapping f of X into itself is called a **contraction** on X if there exists a real r with $٠ \leq r < ١$ such that

d(f(x), f(y)) ≤ rd(x, y),

for all points x and y in X such r is called a **contraction modulus** of f.■

For example, the real valued function

$$f(x) = x^{٢} - ٣x + ٤$$

has ٢ as a fixed point, because f(٢) = ٢.■

It is well known that (as a corollary of Intermediate Value theorem) for any continuous function that map [٠, ١] into itself has a fixed point.■

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable real function. If there is a real number $\beta < ١$ for which the derivative f' satisfies $|f'| \leq \beta$ for all $x \in \mathbb{R}$, then f is a contraction with respect to the usual metric on $\mathbb{R}$ and β is a modulus of contraction for f. ■

Note that:

Not all functions have fixed points: for example, if f is a function defined on the real numbers as f(x) = x + ١, then it has no fixed points, since x is never equal to x + ١ for any real number.

Fixed point theory is divided into three major areas:

١. Topological Fixed Point Theory

٢. Metric Fixed Point Theory

٣. Discrete Fixed Point Theory

Historically the boundary lines between the three areas were defined by the discovery of three major theorems:

١. Brouwer's Fixed Point Theorem

٢. Banach's Fixed Point Theorem

٣. Tarski's Fixed Point Theorem

(In this thesis, we will focus mainly on the second area "Metric Fixed Point Theory").

In this thesis, which consists of four chapters, we obtain some fixed point theorems for **K+١** weakly compatible mappings in a Banach space and example in the case of three functions, Also we find coincidence point for several functions in Banach spaces.

In **Chapter (١)** we give a historical note about fixed point theory, also we give some basic definitions, theorems and lemmas about fixed point and coincidence point theorems. Also, we introduce some different types of contraction such as **an f-contraction, a generalized f-contraction and contractions of the Riech-Rus type etc.** and its related theorems.

In **Chapter (٢)** we study some fixed point and coincidence point in different theorems and their proof that they find fixed point and coincidence point by different ways.

In **Chapter (٣)** we study some applications about fixed point theory.

Finally, in **Chapter (٤)** we give our main results which consist of new theorems about finding common fixed points for several functions by new different ways. We also give an example for the case of three functions.

Chapter ٢

Coincidence fixed point theorems for hybrid mappings

In the present chapter we study some fixed points and coincidence point in different theorems and there proof that they are useful to find fixed point and coincidence point by different ways.

٢.١ Fixed point theorems for compatible multi-valued and single-valued mappings

Hideaki Kaneko [١٦] proved the following result:

Theorem ٢.١,١ [١٦]:

Let (X,d) be a complete metric space, f: X →X and T: X →CB(X) be compatible continuous mappings such that T(X) ⊆f(X) and

$$H(T(x),T(y)) \leq h \max\{d(f(x),f(y)), d(f(x),T(x)), d(f(y),T(y)), \tfrac{١}{٢}[d(f(x),T(y)) + d(f(y),T(x))]\} \qquad (٢.١,١)$$

for all x,y in X, where $٠ \leq h < ١$. Then there exists a point $t \in X$ such that $ft \in Tt$.

Proof

Let $x_٠$ arbitrary point in X. Since $Tx_٠ \subseteq f(X)$, there exists a point $x_١ \in X$ such that $fx_١ \in Tx_٠$. If $h=٠$ then $d(f(x_١), T(x_١)) \leq H(T(x_٠), T(x_١)) = ٠$ i.e $fx_١ \in Tx_١$ since $Tx_١$ is closed. Now assume that $k = \frac{١}{\sqrt{h}} > ١$, by definition of H, there exists a point $y_١ \in Tx_١$ such that $d(y_١, f(x_١)) \leq kH(T(x_١), T(x_٠))$. Since $Tx_١ \subseteq f(X)$, let $x_٢ \in X$ be such that $y_١ = fx_٢ \in Tx_١$.In general x_n have been selected choose $x_{n+١} \in X$ so that $y_n = fx_{n+١} \in Tx_n$, and

$$d(y_n, f(x_n)) \leq kH(T(x_n), T(x_{n-١})),$$

for each $n \geq ١$.

Then using (٢.١.١), we get

$$H(T(x), T(y))$$
$$\leq h \max \{d(f(x), f(y)), d(f(x), T(x)), d(f(y), T(y)),$$
$$\tfrac{١}{٢}[d(f(x), T(y)) + d(f(y), T(x))]\}$$

$$d\big(f(x_{n+1}), f(x_n)\big) \leq kH\big(T(x_{n-1}), T(x_n)\big)$$

$$\leq \sqrt{h}\max\Big\{d\big(f(x_{n-1}), f(x_n)\big), d\big(f(x_{n-1}), T(x_{n-1})\big), d\big(f(x_n), T(x_n)\big),$$

$$\frac{1}{2}\big[d\big(f(x_{n-1}), T(x_n)\big) + d\big(f(x_n), T(x_{n-1})\big)\big]\Big\}$$

Since $fx_n \in Tx_{n-1}$, then $d\big(f(x_n), T(x_{n-1})\big) = 0$,

$$d\big(f(x_{n+1}), f(x_n)\big) \leq \sqrt{h}$$

$$\max\Big\{d\big(f(x_{n-1}), f(x_n)\big), d\big(f(x_{n-1}), f(x_n)\big), d(f(x_n), f(x_{n+1})),$$

$$\frac{1}{2}\big[d(f(x_{n-1}), f(x_{n+1})) + d\big(f(x_n), f(x_n)\big)\big]\Big\}$$

$$d\big(f(x_{n+1}), f(x_n)\big) \leq \sqrt{h}$$

$$\max\Big\{d\big(f(x_{n-1}), f(x_n)\big), d(f(x_n), f(x_{n+1})), \frac{1}{2}\big[d(f(x_{n-1}), f(x_n))$$

$$+ d\big(f(x_n), f(x_{n+1})\big)\big]\Big\}$$

$$d\big(f(x_{n+1}), f(x_n)\big)$$

$$\leq \sqrt{h}\max\{d\big(f(x_{n-1}), f(x_n)\big), d(f(x_n), f(x_{n+1}))\}$$

i.e $d\big(f(x_{n+1}), f(x_n)\big) \leq \sqrt{h}d\big(f(x_n), f(x_{n-1})\big)$ for each n.

Since $\sqrt{h} < ١$, then $\{fx_n\}$ is a Cauchy sequence, hence it converges to some point $t \in X$ using the completeness of X.

Furthermore, the above inequalities also show that

$$kH\big(T(x_{n-١}), T(x_n)\big) \le \sqrt{h}d\big(f(x_n), f(x_{n-١})\big)$$

$$\text{Or } H\big(T(x_{n-١}), T(x_n)\big) \le hd\big(f(x_n), f(x_{n-١})\big).$$

Since $\{fx_n\}$ is Cauchy, this must imply that $\{Tx_n\}$is a Cauchy sequence in the complete metric space $(CB(X), H)$.

Now let $Tx_n \to M \in CB(X)$. Thus

$$d(t, M) \le d\big(t, f(x_n)\big) + d(f(x_n), M)$$
$$\le d\big(t, f(x_n)\big) + H(T(x_{n-١}), M).$$

But $d\big(t, f(x_n)\big) \to ٠$ and $H(T(x_{n-١}), M) \to ٠$ as $n \to \infty$ since M is closed, then $t \in M$ and the compatibility of f and T implies that $H\big(Tf(x_n), fT(x_n)\big) \to ٠$ as $n \to \infty$. This along with the continuities of f and T imply that,

$$d(ft, Tt) \le d\big(ft, ff(x_{n+١})\big) + d(ff(x_{n+١}), Tt)$$
$$\le d\big(ft, ff(x_{n+١})\big) + H(fT(x_n), Tt)$$

$$\leq d\big(ft, ff(x_{n+١})\big) + H\big(fT(x_n), Tf(x_n)\big)$$
$$+H(Tf(x_n), Tt),$$

and the right hand side → ٠ as n → ∞,
i.e. ft ∈ Tt since Tt is closed. ■

٢.٢ A coincidence point theorem for multi-valued contractions

Duran Türkoğlu,Oran Özer and Brian Fisher [٨] proved the following result:

Theorem ٢.٢,١ [٨]:

Let (X, d) be a complete metric space. Let f, g : X → X be continuous mappings and S, T : X → CB(X) be H-continuous mappings such that T (X) ⊆f(X) and S(X) ⊆g(X), Also the pair S and g are compatible mappings, the pair T and f are compatible mappings and

$$H^p(Sx, Ty)$$
$$\leq \max\{ad(fx, gy)d^{p-١}(fx, Sx), ad(fx, gy)d^{p-١}(gy, Ty),$$

$$ad(fx, Sx)d^{p-١}(gy, Ty), cd^{p-١}(fx, Ty)d(gy, Sx)\}, \qquad (٢.٢.١)$$

for all $x, y \in X$, where $٠ < p < ٢$, $٠ < a < ١$ and $c \geq ٠$. Then there exists a point $z \in X$ such that $fz \in Sz$ and $gz \in Tz$, i.e., z is a coincidence point of f, S and of g, T. Further, z is unique when $٠ < c < ١$.

Proof

Let $x_٠$ be an arbitrary point in X. Since $Sx_٠ \subseteq g(X)$, there exists a point $x_١ \in X$ such that $gx_١ \in Sx_٠$ and so there exists a point $y \in Tx_١$ such that

$$d(gx_١, y) \leq kH(Sx_٠, Tx_١),$$

where $k = a^{-١/٢} > ١$ which is possible by Lemma (١.١). Since $Tx_١ \subseteq f(X)$, there exists a point $x_٢ \in X$ such that $y = fx_٢$ and so we have

$$d(gx_١, fx_٢) \leq kH(Sx_٠, Tx_١).$$

Similarly, there exists a point $x_٣ \in X$ such that $gx_٣ \in Sx_٢$ and

$$d(gx_٣, fx_٢) \leq kH(Sx_٢, Tx_١).$$