LAGRANGE GEOMETRY AND GEOMETRIZATION PHILOSOPHY

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Abstract

In the present work, we first study some aspects of the conventional Absolute Parallelism (AP-) geometry. Different curvature tensor fields of the different connections admitted by the AP-space (the metric, dual and symmetric connections) are expressed in a compact form in terms of the torsion tensor field of the canonical connection. The Bianchi identities are applied to simplify some of the formulae obtained and to reveal some of the properties satisfied by a significant fourth order tensor field, which we refer to as the W-tensor. A special case of the canonical connection, namely, being semi-simple, is also investigated. We then construct the Extended Absolute Parallelism (EAP-) geometry, a generalized version of conventional AP-geometry in the double tangent bundle $\pi : TTM \to TM$. EAP-geometry combines, within its structure, the geometric richness of the tangent bundle and the mathematical simplicity of AP-geometry. Consequently, it may have a potentially wider geometric and physical scope than the conventional AP-geometry. All geometric objects considered in the EAP-geometry are not only functions of the positional argument x, but also depend on the directional argument y. Many new geometric objects, which have no counterpart in the conventional AP-geometry, emerge in this different framework. Four different d-connections are introduced: the natural metric, the canonical, the dual and the symmetric d-connections, and their curvature tensor fields (together with their admissible contractions) are computed. The different W-tensors are also obtained. Further conditions are imposed on the canonical d-connection which are assumed to be respectively of Cartan- and Berwald-type. The consequences of these assumptions are investigated. The condition under which the EAP-geometry reduces to the conventional AP-geometry is given. As a physical application of the EAP-geometry, a unified field theory of gravitation and electromagnetism is formulated in the EAP-context. The constructed field theory is a generalization of the Generalized Field Theory (GFT) formulated by Mikhail and Wanas. The theory obtained is purely geometric. The horizontal (resp. vertical) field equations are derived by applying the Euler-Lagrange equations to an appropriate horizontal (resp. vertical) scalar Lagrangian. The symmetric part of the resulting horizontal (resp. vertical) field equations gives rise to a generalized form of Einstein's field equations in which the horizontal (resp. vertical) energy-momentum tensor is purely geometric. The skew-symmetric part of the resulting horizontal (resp. vertical) field equations gives rise to a generalized form of Maxwell's equations in which the electromagnetic field is, again, purely geometric. Some special cases, which reveal the role of the nonlinear connection in the obtained field equations, are studied. The horizontal field equations under the Berwald-type condition are shown to coincide with those of the GFT and the condition under which the constructed field theory reduces to the GFT is explicitly established. A linearization of the obtained field equations, in which a physical interpretation of the coordinates (x^μ, y^a) is discussed, shows that the horizontal field equations in the Cartan-type case are identical to those of the GFT in the first order of approximation. Finally, the approximate solutions of the vertical field equations reveal a wave-like nature (in the directional argument y).

CONTENTS

0. INTRODUCTION

Introduction

"I hold in fact (1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them (2) That this property of being curved or distorted is continually being passed on from one portion of space to another in the manner of a wave (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter." William Clifford as quoted in [48].

"One does not get an answer to the question, What is the state after an atomic collision? but only to the question, How probable is a given effect of the collision? From the standpoint of our quantum mechanics, there is no quantity which causally fixes the effect of a collision in an individual attempt. Should we hope to discover such properties later ... and determine them in individual events? ... I myself am inclined to renounce determinism in the atomic world, but this is a philosophical question for which physical arguments alone do not set standards." Max Born as quoted in [50].

Geometrization and **quantization** philosophies are the main two ideas that dominated fundamental physics during the past century. Geometrization of physics was initiated by Albert Einstein at the beginning of the twentieth century. By introducing space and time as dynamic players, Einstein provided a clear conceptual framework of how gravity works, thus completely overturning our understanding of space and time. Moreover, Einstein has specified the mechanism by which gravity is transmitted: the wrapping of the spacetime fabric.

The geometrization philosophy could be summarized in the following statement: "*To understand nature, one should start with geometry and end with physics*" [64]. Quantization of geometry, starting with spacetime, implies a probability of finding minimum units of length and time in the chosen geometry. [1] In essence, the geometrization philosophy is based on the notion of "continuity", while the quantization philosophy is based on the notion of "discreteness".

The theoretical tools of quantum mechanics and general relativity allow us to understand and make testable predictions about physical happenings from atomic and subatomic realms all the way through phenomena occurring on the scale of galaxies, cluster of galaxies and beyond to the structure of the whole Universe.

[1] In quantum physics, what we observe is not nature itself, but rather *nature exposed to our methods of questioning.* An electron in one context (experimental framework) may act like a particle and in another context it may act like a wave. The essence of the electron is the sum total of all its manifestations (revealed in different contexts). It is not only a wave, it is not only a particle. It is in some sense both and neither. We actually create the required property by choosing the suitable experimental context. In this sense, quantum mechanics does not tell you what is, but what might be!

On the one hand, general relativity and quantum mechanics are **irreconcilable**.[2] Despite more than half a century of hard work by some of the world's leading physicists, these two theories have stubbornly refused to be reconciled, but continue to co-exist in paradoxical and incompatible ways. The notion of a smooth spatial geometry, the central principle of general relativity, is destroyed by the violent fluctuations of the quantum world on short distance scales. On ultramicroscopic scales, the central feature of quantum mechanics - the uncertainty principle - is in direct conflict with the central feature of general relativity - the smooth geometric model of spacetime. In relativity theory, properties of space continue unchanged to smaller and smaller scales: space can be divided and subdivided right down to the dimensionless point, with all properties changing smoothly from point to point. Quantum theory, however, says that such a division is impossible, thus denying the continuity of space and the ultimate reality of the dimensionless point. The reason can be seen in Heisenberg's uncertainty principle, which states that the energy confined within smaller and smaller regions of spacetime becomes increasingly uncertain.[3] A point in the micro-world acquires a "fuzzy" existence and cannot be accurately defined or pinned down and must be thought of as fundamentally non-local, with a smeared out appearance. Indeed, points in spacetime can be thought of as being defined through the collisions and interaction of quantum particles. But the elementary particles themselves are born in collision and quantum interactions of other particles; they live out their lives until they either disintegrate or are swallowed up in another collision. An elementary particle is essentially a history in spacetime, a world line. In this sense, it acquires a "non-local" description.

On the other hand, the two theories are **mutually dependent**. General relativity is a theory about the structure of spacetime, being curved by the amount of matter and energy present. But matter and energy are quantum mechanical in nature, so a complete account of spacetime geometry cannot ignore the quantum nature of matter and energy that creates its very form. Moreover, the interpretations of quantum processes rely upon aspects of relativistic theory. To obtain a definite outcome to a quantum process, it is necessary to use experimental apparatus at the human scale. But since this human scale apparatus comes within the domain of relativity, the experiments used to define quantum states must have a relativistic description. Conversely, the practical way of determining the geometry of spacetime is to survey it using light rays and clocks. But light is quantum mechanical in nature, and the most accurate clocks are atomic clocks. To make precise

[2] If we keep the focus on the attempt to unify relativity and quantum theory, then we are continually impressed by the fact that each of these are transitional theories. Each radically challenges the Newtonian conception of the Universe, but only in part. Each holds unchanged a certain, but different, part of the classical picture. Relativity denies the absolute nature of the Newtonian world, retaining its deterministic aspect, while quantum theory abolishes the deterministic aspect of the classical world, retaining the absoluteness of the Newtonian reality. So the situation is genuinely confusing.

[3] Heisenberg's uncertainty tells us that the shorter a particular time interval, the more uncertain the energy of the system during this interval. For very short intervals - equivalent to the collision times of high energy particles, this energy uncertainty may be very large, large enough to create a new particle. Consequently, new particles are brought into existence for short time periods, only to die back again and release their energy. It is as if quantum theory allows nature to borrow energy and then pay it back, to create and then destroy particles, provided that the sum of energy balances at the end of the collision. It is these so-called "virtual" particles that play the key role in interactions.

measurements of spacetime geometry, we must rely upon systems that are fundamentally quantum mechanical. Relativity again leans on quantum theory.

Four interactions are used to interpret any phenomena known in our universe, so far. These interactions are, gravity, electromagnetism, weak and strong interactions. In the second half of the previous century, several attempts have been made to unify these four interactions using the **quantization** philosophy. This was partially achieved in The Standard Model of Glashow, Weinberg and Salam [9], which represents a unification of the weak and electromagnetic interactions.[4] Physicists had thus been able to unify two of the fundamental forces of nature in terms of gauge fields whose underlying symmetry is broken at low energies. [5] The strong interaction was also included in this scheme, in the less successful attempt, the so called Grand Unified Theories [9].[6] However, it seems very difficult or probably impossible to include gravity with the other interactions using the conventional quantization procedure. In quantizing general relativity, perturbation methods, which have proved to be effective in quantum electrodynamics and Yang-Mills theories, fail in the context of general relativity [24].

The **geometrization** philosophy, on the other hand, is built on the following basis [62]:

- A certain geometry is chosen to represent some model of nature, wide enough to include all required physical objects.
- The chosen geometry should be affinely connected in order to guarantee general covariance of mathematical expressions.
- A one to one correspondence exists between geometric objects and physical entities.
- Differential identities in the chosen geometry represent laws of nature (conservation laws).
- Curves or paths represent the trajectories of test particles and photons.
- Matter should not be represented by a phenomenological material-energy object, but should be built from the geometric objects of the geometric structure.
- Different interactions should not be regarded as logical distinct entities, if any.

[4] Glashow, Weinberg and Salam showed, in essence, that at high enough energy and temperature - such as that occurred a mere fraction of a second after the big bang - electromagnetic and weak force fields **dissolve** into one another, taking on indisinguishable characteristics, and are more accurately called electroweak fields. When the temperature drops, as it has done steadily since the big bang, the two forces **crystallize** out in a different manner from their high common high-temperature form to appear distinct as they appear now in the cold universe we currently inhabit.

[5] The result was a single gauge field which had four components, therefore four vector bosons, which carry the force. Two of these gauge particles have electrical charges, two do not. One of the massless vector bosons is nothing but the photon, the carrier of the electromagnetic force [48].

[6] Essentially there would be a single force within the atomic world, a gauge force carried by massless vector bosons. The new symmetry of this unified force would be made by combining $SU(3)$ - corresponding to the gluon (strong) force - with $SU(2) \times U(1)$ for the electroweak force. The product of these two symmetries gives the required unified force [48].

General relativity meets these requirements perfectly. The geometry chosen for the theory is Riemannian geometry, which is just sufficient to describe gravity. The building block of the geometry is the metric tensor alone. All other geometric objects are constructed using the metric tensor. The affine connection is defined by the Christoffel symbols so that the Riemannian space is affinely connected. The Bianchi identity represents conservation law. Geodesics and null-geodesics represent trajectories of test particles and photons respectively. Finally, there is no phenomenological material energy imposed upon the theory, and the sources of gravitational field arise as constants of integration of the field equations. The last criterion is not applicable since the theory deals solely with gravitational interaction.

This allows us to say that classical general theory of relativity is a complete theory. It prescribes not only the equations which govern the gravitational field, but also the motion of bodies under the influence of this field.

Although the general theory of relativity, constructed in a 4-dimensional Riemannian space, is the best known theory for studying gravitational interactions, so far, it suffers from some problems. Examples of these problems are: the horizon problem, the initial singularity, the flatness of the rotation curve of spiral galaxies [54], the Pioneer 10, 11-anamoly [46] and the interpretation of supernovae type-Ia observation [15]. Some of these problems are old, while others have been discovered in the last ten years or so. In the context of orthodox general relativity theory, there are no satisfactory solutions for such problems.

The above mentioned problems may be due to a missing interaction that has no representative in Riemannian geometry. This may imply that Riemannian geometry is inadequate for studying such problems, since it is limited to the case of a symmetric linear connection and a symmetric metric tensor. However, some authors have suggested modifications of general relativity, retaining Riemannian geometry, by either:

(a) Increasing the order of the Lagrangian used to construct the field equations in general relativity [14].

(b) Using non-conventional equations of state [10].

(c) Adding a term to the theory (i.e the cosmological term) preserving conservation [8].

(d) Adding a term to the theory violating conservation [45].

(e) Increasing the dimension of Riemannian space used (Kaluza-Klien-type theories) [47].

As a result of these varied efforts to modify the general theory of relativity, within the context of Riemannian geometry, we have a variety of spaces leading to a variety of field equations where each set of field equations claiming to represent a generalization of the theory. In other words, we have, in fact, not one generalized theory, but a set of theories, each of which is formulated for a particular purpose.

In formulating general relativity (and possible generalizations), other authors prefer to use more general geometric structures, other than Riemannian geometry, e.g. Riemann-Cartan geometry [13], Absolute Parallelism (AP-) geometry (cf. [33], [63]), Finsler geometry and its generalizations (cf. [2], [3], [31], [52]), special Finsler spaces (cf. [31], [53], [76]), generalized Lagrange spaces (cf. [40]) and generalized AP-geometry [68].[7] The use of such general geometric structures has the advantage of probing the role of geometric entities other than the curvature, e.g. non-symmetric linear connection and its torsion, in physical applications. This may illuminate the role of such entities in physical phenomenae which have no satisfactory interpretation in the context of orthodox general relativity. Moreover, in dealing with the general theory of relativity in the context of Finsler and generalized Lagrange geometry, the very structure of spacetime is actually modified by the dependence of the metric on the direction.

As an example, the use of conventional AP-geometry is found to have the following **geometric** advantages and **physical** achievements:

(a) AP-geometry admits at least four built-in (natural) affine connections, two of which are non-symmetric and three of which have non-vanishing curvature. AP-geometry also admits tensors of third order, a number of second order skew and symmetric tensors and a non-vanishing torsion [32].[8]

(b) AP-geometry has a number of path equations in which the effect of torsion appears explicitly [62].

(c) Associated to an AP-space, there is a Riemannian structure defined in a natural way; thus the AP-space contains within its geometric structure all the mathematical machinery of Riemannian geometry. This facilitates comparison between the results obtained in the context of AP-geometry with the classical relativity theory based on Riemannian geometry [33].

[7] There are other approaches in dealing with the general theory of relativity, which are not only geometric, that exist in the current literature. Istvan Nemati and Hajnal Andreka, the Hungarian Professors at the mathematical Institute of Hungarian Academy of Sciences, together with their student Judit Madarasz, formulated special theory of relativity in an axiomatic method based on a transparent relatively simple set of axioms. Towards general relativity, they played with models where observers are allowed to accelerate. At the limit, using ultra-products or nonstandard analysis, they obtained two notions which are two sides of the same coin. One is **gravity**, or rather the logistic formulation of a four dimensional manifold (**curved** spacetime) which is basically a geometry in the very broad sense of the word. The other is **acceleration** viewed as a delicate patching of instantaneous inertial frames. The **duality** between gravity and acceleration, in short the **equivalence principle**, is thus formulated as a typical **adjoint situation** that pops up in different parts of mathematics [29]. This approach to general relativity uses the machinary of algebraic logic in formulating Einstein's field equations [30]. Consequently, the approach is both **logical** and geometric. The goal of this project is to prove strong theorems of relativity from a small number of easily understandable, convincing axioms. The authors try to elliminate all tacit assumptions from relativity and replace them with explicit and crystal clear axioms in the spirit initiated by Tarski in his first-order axiomatization of geometry [57].

[8] It has been shown that torsion is necessary to **couple** Dirac field to gravity. Moreover, it appears that gauge formulation of gravity needs non-vanishing torsion [17].

(d) Calculations within the framework of AP-geometry are comparable to Riemannian geometry in its simplicity. This is because the basic geometric objects of the AP-space (canonical connection, torsion, contortion, ...) are expressed in a simple form in terms of the vector fields forming the parallelization [33].

(e) Electromagnetism can be successfully represented together with gravity [34].

(f) The use of AP-geometry has helped in solving some of the problems of general relativity, e.g. the problem of localization of gravitational energy [44].

(g) In four dimensions, the tetrad vector field defining the geometric structure of AP-space is used as fundamental variables in an attempt to quantize gravity [18].

(h) AP-geometry gives rise to a new interaction between the torsion of the background geometry and the spin of the moving particles [61], which has been confirmed experimentally [67].

(i) It has been shown that there is a built-in quantum properties in any geometric structure with non-vanishing torsion [66].

(j) In the AP-space, the energy-momentum tensor of fermions can be coupled into the field equations, both its symmetric and skew-symmetric parts. The spacetime structure is determined by matter not only through the symmetric energy-momentum tensor of bosons and the symmetric part of that of fermions, but also through the skew-symmetric part of that of fermions [49].

(k) In four dimensions, using the tetrad vector field, one can always associate a set of scalars with each tensor field defined in the AP-geometry [33].

We consider the Generalized Field Theory (GFT) [34], constructed in the conventional AP-context, as a good example of using geometries with simultaneously non-vanishing curvature and torsion for the purpose of unifying fundamental interactions. Applications of this theory show to what extent it is successful in unifying gravity with electromagnetism (e.g. [59], [60], [65]).[9]

The use of Riemannian geometry in applications explores the role played by its symmetric linear connection (and its consequences) in physical interactions. Similarly, the use of more general geometric structures, with non-vanishing torsion, explores the role played by the non-symmetric linear connection (and its consequences) in physical applications.

[9] Another example of a modified version of general relativity in the context of AP-geometry is that proposed by Moller [44]. In 1978, Moller constructed a field theory using the AP-space for its structure. His aim was to get a theory free from singularities while retaining all the satisfactory features of general relativity. Moller derived the field equations of his theory from a Lagrangian function. Assuming, *a priori*, that the field equations should not involve derivatives higher than the second order partial derivatives of the tetrad vector field, he took the Lagrangian density to be constructed from the tetrad vector field and its first derivative. Moller pointed out [44] that the arbitrariness in the choice of the Lagrangian is decisively limited by the essential requirment that his theory should give the same results of the general theory of relativity inside the solar system.

It is one of the aims of the present work to explore the role of the nonlinear connection in physical phenomena, if any. As a first step to achieve our goal is to construct a field theory in spaces equipped with a nonlinear connection. For this to be done, we have constructed a geometric structure called Extended Absolute Parallelism (EAP-) geometry [78], characterized by a nonlinear connection. EAP-geometry combines, within its structure, the richness of the geometry of the tangent bundle (cf. [23], [72], [74]) and the mathematical simplicity of AP-geometry (cf. [33]). Consequently, *it may have a potentially wider goemetric and physical scope than AP-geometry.* We then construct a generalized version of the GFT within the context of EAP-geometry by performing a suitable extension of the scheme followed in the construction of the field equations of the GFT.

The main purpose of this thesis is fourfold:

(a) A deep study of geometric objects of conventional AP-geometry.

(b) An extension of conventional AP-geometry to geometries of richer structure.

(c) A construction of a unified field theory in this wider framework.

(d) An investigation of the role of the nonlinear connection in the construced theory.

Briefly, in this work, we first study curvature tensor fields together with the torsion tensor fields in the context of conventional AP-geometry. We then give a generalization of conventional AP-geometry in the double tangent bundle $\pi : TTM \to TM$, where TM is the tangent bundle of M. We refer to this geometry as an Extended Absolute Parallelism (EAP-) geometry. As a physical application of the EAP-geometry, a unified theory of gravitation and electromagnetism is formulated in the framework of EAP-geometry on the tangent bundle TM of M. The constructed field equations are a generalization of the GFT. The field equations are derived using a variational technique. Finally, a linearization of the obtained field equations is carried out.

In more detail, the thesis consists of five chapters.

In chapter 1, following the introduction, we focus on the fundamental concepts that will be needed throughout the thesis. We first give a brief account of the conventional Absolute Parallelism geometry, followed by a short survey of the Generalized Field Theory. The geometry of the tangent bundle is then discussed concentrating on the notion of a nonlinear connection, basic for the afterwards developments.

In chapter 2, we investigate curvature tensor fields in the context of conventional Absolute Parallelism (AP-) geometry. Different curvature tensor fields of the different connections admitted by the AP-space are expressed in a compact form in terms of the torsion tensor field of the canonical connection solely. Using the Bianchi identities, some other interesting identities are derived from the expressions obtained. These identities, in turn, are used to simplify some of the relations derived and to reveal some of the properties satisfied by an intriguing fourth order tensor field which we refer to as the W-tensor. Finally, a further condition on the canonical connection is imposed, assuming