Ain Shams University
Faculty of Science
Mathematics Department

STUDY ON SOME TOPOLOGICAL CONCEPTS VIA IDEALS AND THEIR APPLICATIONS

A Thesis
Submitted to Department of Mathematics, Faculty of Science, Ain Shams University for the Degree of Doctor of Philosophy (Ph.D) in Pure Mathematics (Topology)

By

RANA BAHJAT ESMAEEL AL-KHAZRAJI
Department of Mathematics, Faculty of Education for Pure Science (Ibn Al-Haitham), Baghdad University, Baghdad-Iraq

SUPERVISORS

Prof. Dr. MOHAMED E. ABD EL-MONSEF
Department of Mathematic, Faculty of Science, Tanta University
Prof. Dr. ABD EL-AZIZ E. RADWAN
Department of Mathematic, Faculty of Science, Ain Shams University
Prof. Dr. ARAFA A. NASEF
Department of Physics and Engineering Mathematic, Faculty of Engineering, Kafr El-Sheikh University
Prof. Ass. Dr. FAYZA A. IBRAHIM
Department of Mathematic, Faculty of Science, Ain Shams University
Prof. Ass. Dr. ESSAM A. S. EL-SEIDY
Department of Mathematic, Faculty of Science, Ain Shams University

2016

Ain Shams University
Faculty of Science
Mathematics Department

Title of the Thesis

STUDY ON SOME TOPOLOGICAL CONCEPTS VIA IDEALS AND THEIR APPLICATIONS

SUPERVISORS

Prof. Dr. MOHAMED E. ABD EL-MONSEF
Department of Mathematic, Faculty of Science, Tanta University

Prof. Dr. ABD EL-AZIZ E. RADWAN
Department of Mathematic, Faculty of Science, Ain Shams University

Prof. Dr. ARAFA A. NASEF
Department of Physics and Engineering Mathematic, Faculty of Engineering, Kafr El-Sheikh University

Prof. Ass. Dr. FAYZA A. IBRAHIM
Department of Mathematic, Faculty of Science, Ain Shams University

Prof. Ass. Dr. ESSAM A. S. EL-SEIDY
Department of Mathematic, Faculty of Science, Ain Shams University

2016

Acknowledgements

Praise to Allah, for providing me the willingness and strength to a accomplish this work.

I would like to express my deep appreciation and gratitude to all my supervisors *Prof. Dr. M. E. ABD EL-MONSEF, Prof. Dr. A. E. RADWAN, Prof. Dr. A. A. NASEF, Prof. Ass. Dr. F.A. IBRAHIM and Prof. Ass. Dr. ESSAM EL-SEIDY*, for their patience and appreciable advice.

Thanks are extended to the Faculty of Science, Ain Shams University for giving me the chance to complete my post graduate study. I'm also very grateful to the head and all staff members of Mathematics Department.

Also, I would like to thank all persons supplied me the useful references and whoever encouraged me especially *Dr. A. I. NASIR*

Finally, I would like to thank my family so all my friends and fellow students.

RANA BAHJAT ESMAEEL

2016

CONTENTS

Summary

Throughout this thesis (X, τ), (Y, σ) and (Z, μ) (or simply X, Y and Z) are topological spaces with no separation axioms are assumed unless explicitly stated. For each A⊆ X, cl_τ(A) (int_τ(A)) denotes the closure (the interior) of A with respect to τ in order to avoid confusion when there exists more than one topology on X, otherwise the form *cl*(A) (*int*(A)) will be used. $\mathcal{P}$(A) the power set of A is the family of all subsets of A.

In 1933, Kuratowski [38] introduced the concept of an ideal on a nonempty set as follows:
An ideal *I* on a nonempty set X is a nonempty collection of subsets of X which satisfies:

(i) A∈ *I* and B⊆ A implies B∈ *I*, (hereditary),
(ii) A∈ *I* and B∈ *I* implies A∪B∈ *I*, (finite additivity).

Also, several authors are interested in this line of study and therefore some sorts of ideals arise as one goes further in mathematics such as the ideal of finite subsets of X, the ideal of nowhere dense sets and the ideal of meager sets. Different types of operators in terms of ideals, compatibility property, compactness modulo an ideal, sets, functions and other concepts were introduced by many topologists.

The concept of the set operator $( )^*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called a local function, has been introduced by Vaidyanathaswamy in 1945 [68].

In 1967 [54] Newcomb defined a set operator $\Psi'(I, \tau): \tau \rightarrow \tau$, on the other hand, in 1986, Natkaniec [53] defined another operator $\Psi(I,\tau):\mathcal{P}(X)\rightarrow\tau$. It was shown in [23] that the operator Ψ' is simply Ψ restricted to τ.

In 1990, Hamlett, Rose and Jankovic' investigated many properties of the set operator ( )* in [24, 26, 31, 32, 61] and the operator Ψ(I,τ) in [23].
Semadeni, in 1963 [64] and others have recognized a crucial property of ideals which was first proved for the ideal of meager sets by Banach in 1930 [13]. This property of ideals has been called

"compatible" by Njastad in 1966 [56], Jankovic' and Hamlett in 1990 [26, 31], it was called super-compact by Vaidyanathaswamy in 1945 [68], "adherence" by Vaidyanathaswamy in 1960 [69] and strong Banach's localization property by Semadeni in 1963 [64].

The concept of compactness with respect to an ideal was first defined by Newcomb in 1967 [54] and has also been studied by Rančin in 1972 [58]. Compactness with respect to an ideal (namely, *I*-compactness) has been studied extensively in [22, 54, 58].

Newcomb [54] also, defined the concepts of countable compactness modulo an ideal.

In 1981, the term of "semi-compactness" was used for the first time by Dorsett [18].

Hamlett, Jancovic' and Rose in 1991 [25] have studied extensively the later concept under the name "countably *I*-compact".

In 1992, A. A. Nasef [46] was introduced and studied the concepts of α-local function, pre-local function, β-local function, θ-local function and δ-local function which were generalizations for the concept of local function.

In 2001, A. A. Nasef [47], generalized the concept of *I*-compactness by using the concept of semi-open set and presented a new concept of compactness, namely, semi-*I*-compactness. Also, he was studied the relations between semi-*I*-compactness, *I*-compactness and semi-compactness.

The concept of soft sets was first introduced by Molodtsov [44] in 1999 as a general mathematical tool for dealing with uncertain objects.

Recently, in 2011, Shabir and Naz [65] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft neighborhood of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties.

Hussain and Ahmad [29] investigated the properties of open (closed) soft, soft neighborhood and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology and will strengthen the foundations of the theory of soft topological spaces.

Kandil et al. [35], introduced a unification of some types of different kinds of subsets of soft topological spaces using the notions of γ-operation. Kandil et al. [37], generalize this unification of some types of subsets of soft topological spaces using the notions of γ-operation to supra soft topological spaces. The notion of soft ideal is initiated for the first time by Kandil et al. [36]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal $(X, \tau, E, \tilde{I})$.

This thesis consists of seven chapters.

Chapter one consists of three sections. In section one; we review some basic definitions, properties and propositions on ideals. Also, some topological concepts in terms of ideals were recalled.

In section two; we presented the basic definition of soft set in mathematics, then the soft set in soft topological spaces with some concepts and properties which will be needed in the sequel.

In section three; we recalled the basic concepts and terminology of the game theory in order to apply it in the topological spaces and in the ideal topological spaces consequentially.

Chapter two consists of two sections. In section one; we recalled the α-local function [1], which is a generalization of the concept of a local function [68], a set operator $(\ )^*: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, by using the notion α-open set [55]. We further study some properties of these types. Some propositions, remarks and examples are offered to explain these concepts.

In section two; we will introduce the set operator $\Psi^{\alpha}(I, \tau)$ and study some properties of it.

Chapter three consists of two sections. In section one; we recall some types of weakly open sets in ideal topological spaces like; αI-open set [28] and $I\alpha$-open set [4]. Also, we introduce new types of weakly open sets like; α_I-open, weakly αI-open and weakly $I\alpha$-open sets. We study some properties of these types and the relationship between these sets.

In section two; we introduce new types of weakly continuous functions on the ideal topological spaces by using the weakly open sets that we introduced in section one of this chapter like; αI-continuous, $w.\alpha I$-continuous, $w.I\alpha$-continuous, strongly αI-continuous, strongly $w.\alpha I$-continuous, strongly $w.I\alpha$- continuous, αI-irresolute, $w.\alpha I$-irresolute and $w.I\alpha$-irresolute function. The relations among these functions are studied, also.

Chapter four consists of two sections. In section one; we will introduce new types of compactness in an ideal topological space by using the concepts of α_I- open, $w.\alpha I$-open and $w.I\alpha$- open sets like; α_I-compactness, αI-compactness, $I\alpha$-compactness, $w.\alpha I$-compactness, $w.I\alpha$-compactness, $c\alpha_I$-compactness, $c\alpha I$-compactness, $cI\alpha$-compactness, $cw.\alpha I$-compactness and $cw.I\alpha$-compactness. So we study the properties of such spaces and the relationships among them and among other known types with many corollaries. We also gave many counter examples to improve the ones which are invalid and put conditions to the invalid direction true.

In section two; we introduce new types of connectedness in ideal topological spaces by using the concepts of α_I-open, $w.\alpha I$-open and $w.I\alpha$-open sets like; $I\star$-hyperconnectedness, $\alpha_I\star$-hyperconnectedness, $w.\alpha I\star$-hyperconnectedness, $w.I\alpha\star$-hyperconnectedness, α_I-separated connectedness, $w.\alpha I$-separated connectedness and $w.I\alpha$-separated connectedness. Some properties of these types are studied.

In chapter five, we have continued to study the properties of soft topological spaces. We introduced new types of soft compactness based on the soft ideal $\tilde{I}$ in a soft topological space (X, τ, E). Also, several of their topological properties are investigated.

Chapter five consists of two sections. In section one; we introduced new concept of weakly soft open sets in soft topological spaces via ideals which is called α_I-open soft set. So, by using this concept we introduced new types of compactness like; soft α_I-compactness, soft α_I-$\tilde{I}$-compactness, soft α-compactness and soft α-$\tilde{I}$-compactness. Some properties of these types are studied, also.

In section two; we introduced new types of countably compactness in soft topological spaces via ideals like; soft countably α-$\tilde{I}$-compactness and soft countably α_I-$\tilde{I}$-compactness. The relations between these types with some properties are studied, also.

Chapter six consists of four sections. In section one; we propound a comment about the Meneger (X) game. We show that player TWO has a winning strategy always per contra that player ONE. So, we define a new game, say G(*C*), by using the same data of the Meneger (X) without any winning strategy for both players in general.

In section two; we introduce new games by using the concepts of ideal topological spaces, like: G(*C*, *I*), $G_D(C, I)$ and $G_O(C, I)$. So, we show some results that explain many conditions to make anyone of players have winning strategy.

In section three; we introduce new games by using some types of covering for ideal topological spaces. Also, the efficacies of some types of ideals on the strategies for players are studied.

In section four; we use the concept of *I*-compact spaces to define new game say G(X, *I*) and study the relationships between this game and the terms of selection principles.

The aim of chapter seven is to offer a point of view about some assertions which were appeared in some articles ago. This chapter consists of two sections. In section one; we discussed some assertions (The collections of all codense sets and the collections of all boundary sets in any topological space) in the paper by Jankovic´ and Hamlett [30] does not represent ideal by counter examples. And so we show that, the note $I \cup \{X\}$ is not a topology on X in general which was introduced by Jankovic´ and Hamlett [31].

In this section, we show that there is a mathematical error of the claimed result of the paper of Friday Ifeanyi Michael **[**21**]**. We show by producing counter example that the result *"For an ideal I on a topological space X, the following are equivalent: (i) I is the minimal ideal on X, that is, I = {Ø}; (ii) the concepts of semi-openness and I-semi-openness are the same"* which appeared in that article **[**21**]** is incorrect. We review some basic definitions and remarks about "semi-open set" in topological spaces and it is generalization in ideal topological spaces. Also, we explain the relationship between these sets.

Finally, every remark, lemma, proposition and corollary which is not referred to by any reference in this thesis is our own.

Some of the main results in this thesis were published in [2, 3, 48, 49, 51, 52 and 57], other results were presented in *The 27th Conference of Topology and its Applications* (21-22 June 2014), Faculty of Science, Ain Shams University, Egypt.

CHAPTER ONE

Preliminary Concepts and Some Results

CHAPTER 1
Preliminary Concepts and Some Results

1.0 Introduction

In this Chapter, we review three different branches in general topology and in game theory in order to mix them follows.

In section one; we review some basic definitions, properties and propositions on ideals in mathematics. Also, some topological concepts in terms of ideals were recalled.

In section two; we present the basic definition of soft set in mathematics, then the soft set in soft topological spaces with some concepts and properties which will be needed in the sequel.

In section three; we recall the basic concepts and terminology of the game theory in order to apply it in the topological spaces and in the ideal topological spaces consequentially.

For elementary concepts in general topology like; topological space, open set, closed set, interior of a set, exterior of a set, closure of a set, boundary of a set, derived of a set, continuous function, open function, closed function, homeomorphism function, equivalent topological spaces, connected topological space, compactness topological space and soon see [5], [9] and [66].

1.1 Ideals and topology

An ideal *I* on a nonempty set X is a nonempty collection of subsets of X this has hereditary and finite additivity. The subject of ideals in topological spaces has been studied by Kuratowski (1933), Vaidyanathaswamy (1945), Freud (1958), Semadeni (1963), Hayashi (1964), Njastad (1966), Newcomb (1967), Rančin (1972), Samuels (1975), Hashimoto (1976), Natkaniec (1986), Kaniewski and Piotrowski (1986). Recently each of Hamlett, Jankovic' and Rose in (1990, 1991) developed the theme of ideals in general topology.

Definition 1.1.1 [38**]:** A nonempty collection I of subsets of a set X is said to be an ideal on X, if it satisfies the following two conditions:

(i) If $A\in I$ and $B\subseteq A$, then $B\in I$.

(ii) If $A\in I$ and $B\in I$, then $A\cup B\in I$.

(iii) An ideal I is called a σ- ideal if it satisfies the following condition: If $\{A_i: i=1,2,\ldots\}\subseteq I$, then $\cup\{A_i: i=1,2,\ldots\}\in I$ (countable additivty).

Remark 1.1.2 [31, 33**]:**

(i) The simplest ideals on a nonempty set X are $\mathcal{P}(X)$ (the improper ideal) and {Ø} or $I_{\{\emptyset\}}$ (the trivial ideal). Then for every ideal I on X, we have $\{\emptyset\}\subseteq I\subseteq \mathcal{P}(X)$.

(ii) If $X\notin I$, then I is called a proper ideal.

(iii) If I is a proper ideal on a nonempty set X, then $\{A\subseteq X: X-A\in I\}$ is a filter. Hence proper ideals are sometimes called dual filters.

(iv) If I and J are two ideals on a nonempty set X, then we have:

(1) $I\cap J$ is an ideal on X.

(2) In general $I\cup J$ is not an ideal on X, but the join $I\vee J=\{E\cup H: E\in I \text{ and } H\in J\}$ is an ideal on X containing I and J.

(v) $I|_A=\{E\cap A: E\in I\}=\{E\in I: E\subseteq A\}$ is the restriction of I on A, where A is a nonempty subset of X and I is an ideal on X. It is easily seen that $I|_A$ is an ideal.

Example 1.1.3: Let X= {a, b, c, d, e}, I= {Ø, {a}, {c}, {a, c}} and J= {Ø, {b}, {d}, {e}, {b, d}, {b, e}, {d, e}, {b, d, e}}. It is clear that $I\cup J$ is not an ideal on X.

Definition 1.1.4: Newcomb **[**54**]** defined A= B [mod I] if $(A-B)\cup(B-A)\in I$, and observed that "=[mod I]" is an equivalence relation. Let us denote by $A\Delta B$ the "symmetric difference" $(A-B)\cup(B-A)$.

Definition 1.1.5 [10, 31]**:** A subset A of a space (X, τ) is said to be

(i) *nwd* (= nowhere dense) in X, if *int*(*cl*(A))= Ø.

(ii) countable, if it is finite or has the same cardinal number.

(iii) meager set (or of first category) if it is a countable union of *nwd* sets.

(iv) dense- in- itself, if $A\subseteq A^d$, that is A is without isolated points, where A^d is the derived set of A.

(v) perfect, if A is dense- in- itself and closed.

(vi) discrete, if $A \cap A^d = \emptyset$.

(vii) closed and discrete, if $A^d = \emptyset$.

(viii) Relatively compact, if $cl(A)$ is compact in (X, τ).

The following collections form important ideals **[**31, 58**]** on a topological space (X, τ):

(i) $\{\emptyset\}$ or $I_{\{\emptyset\}}$: the trivial ideal.

(ii) $\mathcal{P}(X)$: the improper ideal.

(iii) $\mathcal{F} = I_f$: the ideal of all finite subsets of X.

(iv) $\mathcal{C} = I_c$: the ideal of all countable subsets of X.

(v) $\mathcal{CD}$: (I_{cd}) the ideal of all closed and discrete sets in (X, τ).
i.e. $I_{cd} = \{A \subseteq X: A^d = \emptyset\}$ where A^d is the derived set of A.

(vi) $\mathcal{N} = I_n$: the ideal of all nowhere dense sets.
i.e. $I_n = \{A \subseteq X: int(cl(A)) = \emptyset\}$.

(vii) $\mathcal{M} = I_m$: the ideal of all first category (= meager) sets.
i.e. $I_m = \{A \subseteq X: A = \cup\{B_i , i \in \mathbb{N}; int(cl(B_i)) = \emptyset\}$.

(viii) $\mathcal{CDF} = I_{CDF}$: the ideal of all closed df- sets, where a subset A of a topological space (X, τ) is called discretely finite (= df- set) if for each $x \in A$, there exists an open set U containing x such that $U \cap A$ is finite.

(ix) I_K: the ideal of relatively compact sets in (X, τ).

(x) $\langle A \rangle = I(A)$: the principal ideal generated by any subset A of a space (X, τ). i. e. $I(A) = \mathcal{P}(A) = \{B \subseteq X: B \subseteq A\}$.

Theorem 1.1.6 [54**]:** Let $f: (X, \tau) \to (Y, \sigma)$ be a function. The following statements are true:

(i) If I is an ideal on X, then $f(I) = \{f(E): E \in I\}$ is an ideal on Y.

(ii) If f is injection and J is an ideal on Y, then $f^{-1}(J) = \{f(B): B \in J\}$ is an ideal on X.

Since ideals are usually defined, by Kuratowski **[**38**]**, as collections of subsets of a nonempty set X, Vaidyanathaswamy **[**68, 69**]** introduced the concepts of *a local function* and *a kuratowski closure operator* which were considered the main-entrance of the concept ideal into the topological spaces.